Post-Newtonian General Relativity and Gravitational Waves. Part III: Spin-Orbit Interaction of Two Spinning Compact Bodies

Piotr Jaranowski

Faculty of Physics, University of Białystok, Poland

School of General Relativity, Astrophysics and Cosmology

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The Spin of a Rotating Body

- The spin of a rotating body is of the order

$$S \sim m R v_{\rm spin}\,,$$

  where m and R denote the mass and typical size of the body, respectively, and $v_{\rm spin}$ represents the velocity of the body's surface.

- We are interested in compact bodies, so

$$R \sim \frac{Gm}{c^2}, \quad \text{and then} \quad S \sim G m^2 \frac{v_{\rm spin}}{c^2}.$$

Nomenclature on PN Spin-Dependent Effects

- Formal counting: PN orders are counted in terms of $1/c$ originally present in the Einstein equations, i.e. the spin variables do not contribute to counting of $1/c$. Then, e.g., spin-orbit effects in EOM start as follows:

$$1\text{PN} + 2\text{PN} + \cdots .$$

- Maximally rotating bodies: $v_{\text{spin}} \sim c \Longrightarrow S \sim \dfrac{Gm^2}{c} = \mathcal{O}(c^{-1})$.

  Spin-orbit (i.e. linear in S) effects in EOM:

$$1.5\text{PN} + 2.5\text{PN} + \cdots ;$$

  spin-spin effects in EOM:

$$2\text{PN} + 3\text{PN} + \cdots .$$

- Slowly rotating bodies: $v_{\text{spin}} \ll c \Longrightarrow S \sim \dfrac{Gm^2 v_{\text{spin}}}{c^2} = \mathcal{O}(c^{-2})$.

  Spin-orbit (i.e. linear in S) effects in EOM:

$$2\text{PN} + 3\text{PN} + \cdots ;$$

  spin-spin effects in EOM:

$$3\text{PN} + 4\text{PN} + \cdots .$$

Words are Better than Numbers

- Just words, no $1/c$ counting:

  leading-order (LO)

  $+$ next-to-leading-order (NLO)

  $+$ next-to-next-to-leading-order (NNLO) $+ \cdots .$

General Remarks and Summary

- Generalized ADM formalism for spinning objects is described in Section 7 of 2018 Schäfer/Jaranowski *Living Reviews in Relativity* article and will not be discussed here.
- We present a novel Hamiltonian formulation of the spin-orbit interaction of two spinning compact bodies, valid to linear order in the spins of the bodies (and to any PN order in the orbital part of the interaction).
- It allows one to derive spin-orbit contribution to the orbital equations of motion without having to solve Einstein's field equations with a spin-dependent energy-momentum tensor. One does not use Tulczyjew's pole-dipole energy-momentum tensor. One does not use the Papapetrou translational equations of motion as well.
- Devised and used by Damour, Jaranowski & Schäfer (2008) to compute leading-order and next-to-leading-order two-body spin-orbit Hamiltonians. Then extended by Steinhoff, Hergt & Schäfer (2008) to compute next-to-leading-order spin(1)-spin(2) Hamiltonian.

3-Dimensional Euclidean Spin Vector (1/2)

- The translational and rotational equations of motion of a spinning particle in curved spacetime, to linear order in the spin, read:

$$m\frac{\mathrm{D}u_\mu}{\mathrm{d}\tau} = \frac{1}{2}\frac{\epsilon^{\alpha\beta\lambda\rho}}{\sqrt{-g}}\tilde{S}_\mu u_\beta u_\nu R^\nu{}_{\mu\lambda\rho},$$

$$\frac{\mathrm{D}\tilde{S}_\mu}{\mathrm{d}\tau} = 0.$$

  Here u^μ is normalized 4-velocity of the spinning particle,

$$u^\mu := c^{-1}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}, \quad u^\mu u_\mu = -1,$$

  τ denotes the proper time parameter along the world line $x^\mu(\tau)$,
  m and $\tilde{S}^\mu$ is mass and 4-dimensional spin vector of the spinning particle,
  D is the 4-dimensional covariant derivative,
  $R^\nu{}_{\mu\lambda\rho}$ the Riemann curvature tensor, $g := \det(g_{\mu\nu})$,
  $\epsilon^{\alpha\beta\lambda\rho}$ is the completely antisymmetric (flat-spacetime) Levi-Civita symbol with $\epsilon^{0123} = 1$.

- We shall not need to consider the translational equations of motion.

3-Dimensional Euclidean Spin Vector (2/2)

- The 4-dimensional length of $\tilde{S}^\mu$ is preserved along the world line:

$$\frac{\mathrm{D}\tilde{S}_\mu}{\mathrm{d}\tau} = 0 \quad \Longrightarrow \quad g^{\mu\nu}\tilde{S}_\mu \tilde{S}_\nu = s^2, \quad s^2 = \text{const.}$$

- Making use of covariant spin supplementary condition

$$\tilde{S}_\mu u^\mu = 0 \quad \Longrightarrow \quad \tilde{S}_0 = -\tilde{S}_i v^i \qquad \left(v^i := c^{-1}\mathrm{d}x^i/\mathrm{d}t\right),$$

  one gets

$$g^{\mu\nu}\tilde{S}_\mu \tilde{S}_\nu = G^{ij}\tilde{S}_i \tilde{S}_j = s^2,$$

  where G^{ij} is a positive-definite symmetric matrix:

$$G^{ij} := g^{ij} - g^{0i}v^j - g^{0j}v^i + g^{00}v^i v^j.$$

- The positive-definite symmetric matrix G^{ij}
  admits a unique positive-definite symmetric square root,
  say $H^{ij} = H^{ji}$, such that

$$G^{ij} = H^{ik}H^{kj}.$$

- One defines a constant-in-magnitude 3-dimensional Euclidean spin vector $S_i \equiv S^i$ as

$$S_i := H^{ij}\tilde{S}_j, \quad S_i S_i = s^2.$$

Spin Precession Equation

- Evolution equation for the spin vector S_i:

$$\frac{\mathrm{d}S_i}{\mathrm{d}t} = V^{ij} S_j, \quad V^{ij} := \frac{\mathrm{d}H^{ik}}{\mathrm{d}t}(H^{-1})^{kj} + H^{ik}\tilde{V}^{kl}(H^{-1})^{lj},$$
$$\tilde{V}^{ij} := c\Big(\Gamma^j{}_{i0} + \Gamma^j{}_{ik}v^k - \Gamma^0{}_{i0}v^j - \Gamma^0{}_{ik}v^j v^k\Big).$$

- The matrix V^{ij} is antisymmetric, $V^{ij} = -V^{ji}$, so one can introduce the 3-dimensional Euclidean pseudo-vector of the angular velocity of rotation of the spin,

$$\Omega_i := -\frac{1}{2}\varepsilon_{ijk}V^{jk} \quad (\text{so } V^{ij} = -\varepsilon^{ijk}\Omega_k).$$

- One gets a Newtonian looking spin precession equation:

$$\frac{\mathrm{d}\mathbf{S}}{\mathrm{d}t} = \mathbf{\Omega} \times \mathbf{S}.$$

Spin-Orbit Hamiltonian

- Variables in phase space of 2 interacting spinning particles:

$$\mathbf{x}_a = (x_a^i), \quad \mathbf{p}_a = (p_{ai}), \quad \mathbf{S}_a = (S_{ai}) \qquad (a = 1, 2, \quad i = 1, 2, 3).$$

  The Poisson brackets relations:

$$\{x_a^i, p_{bj}\} = \delta_{ab}\delta_j^i, \quad \{S_{ai}, S_{bj}\} = \delta_{ab}\varepsilon_{ijk}S_{ak}, \quad \text{zero otherwise.}$$

- One looks for a Hamiltonian of the general form:

$$H(\mathbf{x}_a, \mathbf{p}_a, \mathbf{S}_a) = H_{\mathrm{o}}(\mathbf{x}_a, \mathbf{p}_a) + H_{\mathrm{so}}(\mathbf{x}_a, \mathbf{p}_a, \mathbf{S}_a),$$

  H_{o} is the orbital part and H_{so} is the spin-orbit part of H.

- One can always write H_{so} in the general form

$$H_{\mathrm{so}}(\mathbf{x}_a, \mathbf{p}_a, \mathbf{S}_a) = \sum_a \boldsymbol{\Omega}_a(\mathbf{x}_b, \mathbf{p}_b) \cdot \mathbf{S}_a,$$

  $\boldsymbol{\Omega}_a = (\Omega_a^i)$ depends on the orbital degrees of freedom $(\mathbf{x}_a, \mathbf{p}_a)$, but does not depend on the spins $\mathbf{S}_a$.

- Such introduced $\boldsymbol{\Omega}_a$ is not only a notation for the coefficient of $\mathbf{S}_a$ in H_{so}, but it is equal to the angular velocity with which the spin vector $\mathbf{S}_a$ precesses. To show this it is enough to compute $\mathrm{d}\mathbf{S}_a/\mathrm{d}t$,

$$\frac{\mathrm{d}\mathbf{S}_a}{\mathrm{d}t} = \{\mathbf{S}_a, H_{\mathrm{so}}(\mathbf{x}_b, \mathbf{p}_b, \mathbf{S}_b)\} = \boldsymbol{\Omega}_a(\mathbf{x}_b, \mathbf{p}_b) \times \mathbf{S}_a.$$

The Arbitrariness in the Definition of the Conserved Spin: a "Gauge Symmetry" [Under a Local SO(3) Group] (1/2)

- The condition $S_i S_i = s^2$ is unchanged, if

$$S_i \to S_i', \quad \text{with} \quad S_i' = R_{ij} S_j,$$

  where R is an arbitrary 3-dimensional Euclidean rotation matrix.

- An infinitesimal rotation,

$$R_{ij} = \delta_{ij} - \theta_{ij},$$

  where θ_{ij} is a small antisymmetric matrix, leads to

$$\delta \mathbf{S} \equiv \mathbf{S}' - \mathbf{S} = \boldsymbol{\theta} \times \mathbf{S}, \quad \theta_i \equiv \frac{1}{2} \varepsilon_{ijk} \theta_{jk}.$$

- An infinitesimal canonical transformation g in the phase space $(\mathbf{x}, \mathbf{p}, \mathbf{S})$ acts on any phase-space function f according to

$$\delta f = \{f, g\}.$$

- A canonical transformation of the form

$$g(\mathbf{x}, \mathbf{p}, \mathbf{S}) := \boldsymbol{\theta}(\mathbf{x}, \mathbf{p}) \cdot \mathbf{S}$$

  transforms the spin vector according to

$$\delta \mathbf{S} = \{\mathbf{S}, g\} = \boldsymbol{\theta} \times \mathbf{S},$$

  which reproduces the effect of an infinitesimal local rotation.

The Arbitrariness in the Definition of the Conserved Spin: a "Gauge Symmetry" [Under a Local SO(3) Group] (2/2)

- The canonical transformation g also acts on the orbital degrees of freedom $(\mathbf{x}, \mathbf{p})$:

$$\mathbf{x} \to \mathbf{x}' = \mathbf{x} + \delta\mathbf{x} = \mathbf{x} + \{\mathbf{x}, g\},$$
$$\mathbf{p} \to \mathbf{p}' = \mathbf{p} + \delta\mathbf{p} = \mathbf{p} + \{\mathbf{p}, g\}.$$

- Dynamics of the system in new variables $(\mathbf{x}', \mathbf{p}', \mathbf{S}')$ is equivalent to original dynamics, but with the Hamiltonian

$$H'\big(\mathbf{x}', \mathbf{p}', \mathbf{S}'\big) := H\big(\mathbf{x}(\mathbf{x}', \mathbf{p}', \mathbf{S}'), \mathbf{p}(\mathbf{x}', \mathbf{p}', \mathbf{S}'), \mathbf{S}(\mathbf{x}', \mathbf{p}', \mathbf{S}')\big).$$

- The corresponding change of the spin angular velocity:

$$\boldsymbol{\Omega} \to \boldsymbol{\Omega}' = \boldsymbol{\Omega} + \frac{\mathrm{d}\boldsymbol{\theta}}{\mathrm{d}t}.$$

Computation of Precession Angular Velocities (1/2)

- One splits the 4-dimensional metric $g_{\mu\nu}$ into 3-dimensional objects $(\alpha, \beta_i, \gamma_{ij})$,

$$\alpha := (-g^{00})^{-1/2}, \quad \beta_i := g_{0i}, \quad \gamma_{ij} := g_{ij},$$

  and one employs the ADMTT coordinate conditions

$$\gamma_{ij} = \left(1 + \frac{1}{8}\phi\right)^4 \delta_{ij} + h_{ij}^{\mathrm{TT}}, \quad \pi^{ii} = 0 \qquad (\pi^{ij} = \tilde{\pi}^{ij} + \pi_{\mathrm{TT}}^{ij}).$$

- After PN expansion of all quantities the angular velocity is decomposed as

$$\Omega_a^i = \Omega_{a(2)}^i + \Omega_{a(4)}^i + \mathcal{O}(c^{-6}),$$

  where

$$\Omega_{a(2)}^i \propto G/c^2 \quad \text{(LO contribution)},$$
$$\Omega_{a(4)}^i \propto G/c^4 + G^2/c^4 \quad \text{(NLO contribution)}.$$

- The LO and NLO angular velocities:

$$\Omega^i_{a(2)}/c = \frac{1}{2}\varepsilon_{ijk}\,\mathrm{Reg}_a\Big\{\beta_{(3)j,k} + \Big(\alpha_{(2),j} - \frac{1}{2}\phi_{(2),j}\Big)v^k_{a(1)}\Big\},$$

$$\begin{aligned}\Omega^i_{a(4)}/c = \frac{1}{2}\varepsilon_{ijk}\,\mathrm{Reg}_a\Big\{&\beta_{(5)j,k} + \beta_{(3)k}\alpha_{(2),j} - \frac{1}{2}\phi_{(2)}\beta_{(3)j,k} + \frac{1}{16}\phi_{(2)}\phi_{(2),j}v^k_{a(1)} \\ &- \frac{1}{2}\phi_{(4),j}v^k_{a(1)} - h^{\mathrm{TT}}_{(4)kl,j}v^l_{a(1)} + (\alpha_{(4),j} - \alpha_{(2)}\alpha_{(2),j})v^k_{a(1)} + \tilde{\pi}^{jl}_{(3)}v^k_{a(1)}v^l_{a(1)} \\ &- \frac{1}{2}\alpha_{(2),k}v^j_{a(1)}v^l_{a(1)}v^l_{a(1)} + \frac{1}{4}\frac{\dot{v}^j_{a(1)}}{c}v^k_{a(1)}v^l_{a(1)}v^l_{a(1)} + \Big(\alpha_{(2),j} - \frac{1}{2}\phi_{(2),j}\Big)v^k_{a(3)}\Big\}.\end{aligned}$$

- Computation relies on:
  - (i) insertion of the explicit form of the 2PN-accurate metric describing two spinless particles (one needs to know $\phi_{(2)}$, $\phi_{(4)}$, $\tilde{\pi}^{ij}_{(3)}$, $h^{\mathrm{TT}}_{(4)ij}$, $\alpha_{(2)}$, $\alpha_{(4)}$, $\beta_{(3)i}$, $\beta_{(5)i}$).
  - (ii) expressing the velocities v^i_a in terms of the canonical variables,
  - (iii) Hadamard's "partie finie" regularization of the self-interaction terms.

The Result: LO and NLO Spin-Orbit Hamiltonians

The LO and NLO angular velocities in terms of $\mathbf{x}_a$ and $\mathbf{p}_a$:

$$\mathbf{\Omega}_{1(2)} = \frac{G}{c^2\, r_{12}^2}\left(\frac{3m_2}{2m_1}\mathbf{n}_{12}\times\mathbf{p}_1 - 2\mathbf{n}_{12}\times\mathbf{p}_2\right),$$

$$\mathbf{\Omega}_{1(4)} = \frac{G^2}{c^4\, r_{12}^3}\left\{\left(-\frac{11}{2}m_2 - 5\frac{m_2^2}{m_1}\right)\mathbf{n}_{12}\times\mathbf{p}_1 + \left(6m_1 + \frac{15}{2}m_2\right)\mathbf{n}_{12}\times\mathbf{p}_2\right\}$$
$$+ \frac{G}{c^4\, r_{12}^2}\left\{\left(-\frac{5m_2\mathbf{p}_1^2}{8m_1^3} - \frac{3(\mathbf{p}_1\cdot\mathbf{p}_2)}{4m_1^2} + \frac{3\mathbf{p}_2^2}{4m_1 m_2} - \frac{3(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{4m_1^2} - \frac{3(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{2m_1 m_2}\right)\mathbf{n}_{12}\times\mathbf{p}_1\right.$$
$$\left. + \left(\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1 m_2} + \frac{3(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1 m_2}\right)\mathbf{n}_{12}\times\mathbf{p}_2 + \left(\frac{3(\mathbf{n}_{12}\cdot\mathbf{p}_1)}{4m_1^2} - \frac{2(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1 m_2}\right)\mathbf{p}_1\times\mathbf{p}_2\right\}.$$

The spin-orbit Hamiltonians to LO and NLO orders:

$$H_{\mathrm{so}}(\mathbf{x}_a,\mathbf{p}_a,\mathbf{S}_a) = \frac{1}{c^2}\,H_{\mathrm{so}}^{\mathrm{LO}}(\mathbf{x}_a,\mathbf{p}_a,\mathbf{S}_a) + \frac{1}{c^4}\,H_{\mathrm{so}}^{\mathrm{NLO}}(\mathbf{x}_a,\mathbf{p}_a,\mathbf{S}_a) + \mathcal{O}(c^{-6}),$$

$$\frac{1}{c^2}\,H_{\mathrm{so}}^{\mathrm{LO}}(\mathbf{x}_a,\mathbf{p}_a,\mathbf{S}_a) = \sum_a \mathbf{\Omega}_{a(2)}(\mathbf{x}_b,\mathbf{p}_b)\cdot\mathbf{S}_a,$$

$$\frac{1}{c^4}\,H_{\mathrm{so}}^{\mathrm{NLO}}(\mathbf{x}_a,\mathbf{p}_a,\mathbf{S}_a) = \sum_a \mathbf{\Omega}_{a(4)}(\mathbf{x}_b,\mathbf{p}_b)\cdot\mathbf{S}_a.$$

The translational equations of motion of two spinning particles:

$$\dot{\mathbf{x}}_a = \frac{\partial H(\boldsymbol{x}_b,\mathbf{p}_b,\mathbf{S}_b)}{\partial\mathbf{p}_a}, \quad \dot{\mathbf{p}}_a = -\frac{\partial H(\boldsymbol{x}_b,\mathbf{p}_b,\mathbf{S}_b)}{\partial\mathbf{x}_a}.$$

Poincaré Algebra

One should prove the existence of ten phase-space generators

$$H(\boldsymbol{x}_a, \mathbf{p}_a, \mathbf{S}_a),\ P_i(\boldsymbol{x}_a, \mathbf{p}_a, \mathbf{S}_a),\ J_i(\boldsymbol{x}_a, \mathbf{p}_a, \mathbf{S}_a),\ G_i(\boldsymbol{x}_a, \mathbf{p}_a, \mathbf{S}_a),$$

whose Poisson brackets reproduce the Poincaré algebra:

$$\{P_i, P_j\} = 0, \quad \{P_i, H\} = 0, \quad \{J_i, H\} = 0,$$

$$\{J_i, P_j\} = \varepsilon_{ijk}\, P_k, \quad \{J_i, J_j\} = \varepsilon_{ijk}\, J_k,$$

$$\{J_i, G_j\} = \varepsilon_{ijk}\, G_k,$$

$$\{G_i, H\} = P_i,$$

$$\{G_i, P_j\} = \frac{1}{c^2}\, H\, \delta_{ij},$$

$$\{G_i, G_j\} = -\frac{1}{c^2}\, \varepsilon_{ijk}\, J_k.$$

Poincaré Algebra Generators

- The translation P_i and rotation J_i generators are realized as

$$P_i(\boldsymbol{x}_a, \mathbf{p}_a, \mathbf{S}_a) = \sum_a p_{ai}, \quad J_i(\boldsymbol{x}_a, \mathbf{p}_a, \mathbf{S}_a) = \sum_a \left(\varepsilon_{ik\ell}\, x_a^k\, p_{a\ell} + S_{ai}\right).$$

- The Hamiltonian $H(\mathbf{x}_a, \mathbf{p}_a, \mathbf{S}_a) = H_{\mathrm{o}}(\mathbf{x}_a, \mathbf{p}_a) + H_{\mathrm{so}}(\mathbf{x}_a, \mathbf{p}_a, \mathbf{S}_a)$, where

$$H_{\mathrm{o}}(\mathbf{x}_a, \mathbf{p}_a) = (m_1 + m_2)c^2 + H_{\mathrm{o}}^{\mathrm{N}}(\mathbf{x}_a, \mathbf{p}_a) + \frac{1}{c^2}\, H_{\mathrm{o}}^{\mathrm{1PN}}(\mathbf{x}_a, \mathbf{p}_a) + \frac{1}{c^4}\, H_{\mathrm{o}}^{\mathrm{2PN}}(\mathbf{x}_a, \mathbf{p}_a) + \mathcal{O}(c^{-6}),$$

$$H_{\mathrm{so}}(\mathbf{x}_a, \mathbf{p}_a, \mathbf{S}_a) = \frac{1}{c^2}\, H_{\mathrm{so}}^{\mathrm{LO}}(\mathbf{x}_a, \mathbf{p}_a, \mathbf{S}_a) + \frac{1}{c^4}\, H_{\mathrm{so}}^{\mathrm{NLO}}(\mathbf{x}_a, \mathbf{p}_a, \mathbf{S}_a) + \mathcal{O}(c^{-6}).$$

- The center-of-mass vector $\mathbf{G}(\mathbf{x}_a, \mathbf{p}_a, \mathbf{S}_a) = \mathbf{G}_{\mathrm{o}}(\mathbf{x}_a, \mathbf{p}_a) + \mathbf{G}_{\mathrm{so}}(\mathbf{x}_a, \mathbf{p}_a, \mathbf{S}_a)$, where

$$\mathbf{G}_{\mathrm{o}}(\mathbf{x}_a, \mathbf{p}_a) = \sum_a m_a \mathbf{x}_a + \frac{1}{c^2}\, \mathbf{G}_{\mathrm{o}}^{\mathrm{1PN}}(\mathbf{x}_a, \mathbf{p}_a) + \frac{1}{c^4}\, \mathbf{G}_{\mathrm{o}}^{\mathrm{2PN}}(\mathbf{x}_a, \mathbf{p}_a) + \mathcal{O}(c^{-6}),$$

$$\mathbf{G}_{\mathrm{so}}(\mathbf{x}_a, \mathbf{p}_a, \mathbf{S}_a) = \frac{1}{c^2}\, \mathbf{G}_{\mathrm{so}}^{\mathrm{LO}}(\mathbf{x}_a, \mathbf{p}_a, \mathbf{S}_a) + \frac{1}{c^4}\, \mathbf{G}_{\mathrm{so}}^{\mathrm{NLO}}(\mathbf{x}_a, \mathbf{p}_a, \mathbf{S}_a) + \mathcal{O}(c^{-6}).$$

- The LO term is known from the special-relativistic limit,

$$\mathbf{G}_{\mathrm{so}}^{\mathrm{LO}}(\mathbf{x}_a, \mathbf{p}_a, \mathbf{S}_a) = -\frac{\mathbf{S}_1 \times \mathbf{p}_1}{2m_1} + (1 \leftrightarrow 2).$$

The Method of Undetermined Coefficients

- The most general form of $\mathbf{G}_{\rm so}^{\rm NLO}$ can a priori depend on 8 unknown dimensionless coefficients $g_1, \ldots, g_8$ $[(V_1, V_2, V_3) := \mathbf{V}_1 \cdot (\mathbf{V}_2 \times \mathbf{V}_3) = \varepsilon_{ijk} V_1^i V_2^j V_3^k]$:

$$
\begin{aligned}
\mathbf{G}_{\rm so}^{\rm NLO} = {} & \frac{\mathbf{p}_1^2}{8m_1^3} \mathbf{S}_1 \times \mathbf{p}_1 \\
& + \frac{Gm_2}{r_{12}} \left\{ g_1 \frac{\mathbf{S}_1 \times \mathbf{p}_1}{m_1} + g_2 \frac{\mathbf{S}_1 \times \mathbf{p}_2}{m_2} + \left(g_3 \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)}{m_1} + g_4 \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_2} \right) \mathbf{n}_{12} \times \mathbf{S}_1 \right. \\
& \left. + \left(g_5 \frac{(S_1, n_{12}, p_1)}{m_1} + g_6 \frac{(S_1, n_{12}, p_2)}{m_2} \right) \boldsymbol{n}_{12} \right\} \\
& + \frac{Gm_2}{r_{12}^2} \left\{ g_7 \frac{(S_1, n_{12}, p_1)}{m_1} + g_8 \frac{(S_1, n_{12}, p_2)}{m_2} \right\} \mathbf{x}_1 + (1 \leftrightarrow 2).
\end{aligned}
$$

- The generators P_i, J_i, H, and G_i in the forms given above exactly satisfy the relations $\{P_i, P_j\} = 0$, $\{P_i, H\} = 0$, $\{J_i, H\} = 0$, $\{J_i, P_j\} = \varepsilon_{ijk} P_k$, $\{J_i, J_j\} = \varepsilon_{ijk} J_k$, $\{J_i, G_j\} = \varepsilon_{ijk} G_k$.
- There exist unique values of the coefficients $g_1, \ldots, g_8$ ensuring the fulfillment of the relations $\{G_i, H\} = P_i$, $\{G_i, P_j\} = \frac{1}{c^2} H \delta_{ij}$, $\{G_i, G_j\} = -\frac{1}{c^2} \varepsilon_{ijk} J_k$.

The Transformation Between the ADM and Harmonic Variables

- The NLO spin-dependent contributions in the translational and rotational equations of motion of two spinning particles, were computed in harmonic coordinates by BBF (Blanchet, Buonanno, and Faye, 2006).
- BBF use two different spin variables:
  spin variables $\mathbf{S}_a^{\mathrm{BBF}}$ with nonconserved Euclidean magnitudes,
  and spin variables $\mathbf{S}_a^{\mathrm{c\,BBF}}$ with conserved Euclidean lengths.
- We look for the explicit transformation between
  the ADM variables $(\boldsymbol{x}_a, \mathbf{p}_a, \mathbf{S}_a)$,
  and the harmonic variables $(\mathbf{y}_a, \mathbf{v}_a := \dot{\mathbf{y}}_a, \mathbf{S}_a^{\mathrm{c\,BBF}})$:

$$\mathbf{y}_a(t) = \mathbf{Y}_a(\mathbf{x}_b(t), \mathbf{p}_b(t), \mathbf{S}_b(t)),$$

$$\mathbf{S}_a^{\mathrm{c\,BBF}}(t) = \boldsymbol{\Sigma}_a(\mathbf{x}_b(t), \mathbf{p}_b(t), \mathbf{S}_b(t)).$$

The Transformation Between Spin Variables

- BBF computed the angular velocity vector $\boldsymbol{\Omega}_a^{\mathrm{BBF}}$,

$$\frac{\mathrm{d}\mathbf{S}_a^{\mathrm{c\,BBF}}}{\mathrm{d}t} = \boldsymbol{\Omega}_a^{\mathrm{BBF}} \times \mathbf{S}_a^{\mathrm{c\,BBF}}, \quad a = 1, 2.$$

- We have reexpressed $\boldsymbol{\Omega}_a^{\mathrm{BBF}}(\mathbf{y}_b, \mathbf{v}_b)$ in terms of ADM variables and compared the result with $\boldsymbol{\Omega}_a(\mathbf{x}_b, \mathbf{p}_b)$. We have found

$$\boldsymbol{\Omega}_{a(2)}^{\mathrm{BBF}}(\mathbf{y}_b, \mathbf{v}_b) = \boldsymbol{\Omega}_{a(2)}(\mathbf{x}_b, \mathbf{p}_b),$$

$$\boldsymbol{\Omega}_{a(4)}^{\mathrm{BBF}}(\mathbf{y}_b, \mathbf{v}_b) = \boldsymbol{\Omega}_{a(4)}(\mathbf{x}_b, \mathbf{p}_b) + \frac{\mathrm{d}\boldsymbol{\theta}_a}{\mathrm{d}t},$$

  where

$$\boldsymbol{\theta}_1 = \frac{G}{c^4 r_{12}}\left(-\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{4m_1}\mathbf{n}_{12}\times\mathbf{p}_1 + \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_2}\mathbf{n}_{12}\times\mathbf{p}_2 - \frac{9}{4m_1}\mathbf{p}_1\times\mathbf{p}_2 \right).$$

- From these results one can deduce that

$$\boldsymbol{\Sigma}_a(\mathbf{x}_b, \mathbf{p}_b, \mathbf{S}_b) = \mathbf{S}_a + \boldsymbol{\theta}_a(\mathbf{x}_b, \mathbf{p}_b)\times\mathbf{S}_a.$$

The Transformation Between Orbital Degrees of Freedom (1/4)

- One can decompose the transformation $\mathbf{Y}_a$ between ADM and harmonic orbital degrees of freedom into spin-independent and spin-dependent terms:

$$\mathbf{Y}_a(\mathbf{x}_b, \mathbf{p}_b, \mathbf{S}_b) = \mathbf{x}_a + \mathbf{Y}_a^{\mathrm{O}}(\mathbf{x}_b, \mathbf{p}_b) + \mathbf{Y}_a^{\mathrm{SO}}(\mathbf{x}_b, \mathbf{p}_b, \mathbf{S}_b),$$

  where the spin-dependent term is of the form

$$\mathbf{Y}_a^{\mathrm{SO}}(\mathbf{x}_b, \mathbf{p}_b, \mathbf{S}_b) = \mathbf{Y}_{a(2)}^{\mathrm{SO}}(\mathbf{x}_b, \mathbf{p}_b, \mathbf{S}_b) + \mathbf{Y}_{a(4)}^{\mathrm{SO}}(\mathbf{x}_b, \mathbf{p}_b, \mathbf{S}_b) + \mathcal{O}(c^{-6}).$$

- The LO spin-dependent part equals

$$\mathbf{Y}_{a(2)}^{\mathrm{SO}}(\mathbf{x}_b, \mathbf{p}_b, \mathbf{S}_b) = \frac{\mathbf{S}_a \times \mathbf{p}_a}{2 m_a^2 c^2}.$$

The Transformation Between Orbital Degrees of Freedom (2/4)

- One determines the NLO spin-dependent part by using the method of undetermined coefficients. One thus consideres the most general template for $\mathbf{Y}^{\rm SO}_{a(4)}$, which depends on 12 unknown coefficients:

$$
\begin{aligned}
\mathbf{Y}^{\rm SO}_{1(4)}(\mathbf{x}_a, \mathbf{p}_a, \mathbf{S}_a) = &-\frac{\mathbf{p}_1^2}{8c^4 m_1^4}\mathbf{S}_1 \times \mathbf{p}_1 + \frac{Gm_2}{c^4 r_{12}}\frac{1}{m_1}\Bigg(a_1 \frac{\mathbf{S}_1 \times \mathbf{p}_1}{m_1} + a_2 \frac{\mathbf{S}_1 \times \mathbf{p}_2}{m_2} \\
&+ \left(a_3 \frac{(\mathbf{n}_{12}\cdot \mathbf{p}_1)}{m_1} + a_4 \frac{(\mathbf{n}_{12}\cdot \mathbf{p}_2)}{m_2}\right)\mathbf{n}_{12} \times \mathbf{S}_1 + \left(a_5 \frac{(S_1, n_{12}, p_1)}{m_1} + a_6 \frac{(S_1, n_{12}, p_2)}{m_2}\right)\boldsymbol{n}_{12}\Bigg) \\
&+ \frac{G}{c^4 r_{12}}\Bigg(b_1 \frac{\mathbf{S}_2 \times \mathbf{p}_1}{m_1} + b_2 \frac{\mathbf{S}_2 \times \mathbf{p}_2}{m_2} + \left(b_3 \frac{(\mathbf{n}_{12}\cdot \mathbf{p}_1)}{m_1} + b_4 \frac{(\mathbf{n}_{12}\cdot \mathbf{p}_2)}{m_2}\right)\mathbf{n}_{12} \times \mathbf{S}_2 \\
&+ \left(b_5 \frac{(S_2, n_{12}, p_1)}{m_1} + b_6 \frac{(S_2, n_{12}, p_2)}{m_2}\right)\boldsymbol{n}_{12}\Bigg).
\end{aligned}
$$

FIRST METHOD

- One can compute the coefficients $a_1, \ldots, a_6$, $b_1, \ldots, b_6$ by comparing the ten conserved quantities derived by BBF in harmonic coordinates,

$$E(\mathbf{y}_a, \mathbf{v}_a, \mathbf{S}_a^{\mathrm{c\,BBF}}), \quad \mathbf{P}(\mathbf{y}_a, \mathbf{v}_a, \mathbf{S}_a^{\mathrm{c\,BBF}}), \quad \mathbf{J}(\mathbf{y}_a, \mathbf{v}_a, \mathbf{S}_a^{\mathrm{c\,BBF}}), \quad \mathbf{G}(\mathbf{y}_a, \mathbf{v}_a, \mathbf{S}_a^{\mathrm{c\,BBF}}),$$

  with the ten phase-space Poincaré generators constructed within Hamiltonian formalism. This is done by making replacements

$$\begin{aligned} \mathbf{y}_a &\to \mathbf{Y}_a(\boldsymbol{x}_b, \mathbf{p}_b, \mathbf{S}_b) \\ \mathbf{v}_a &\to \mathbf{V}_a(\boldsymbol{x}_b, \mathbf{p}_b, \mathbf{S}_b) = \{\mathbf{Y}_a(\boldsymbol{x}_b, \mathbf{p}_b, \mathbf{S}_b), H(\boldsymbol{x}_b, \mathbf{p}_b, \mathbf{S}_b)\} \\ \mathbf{S}_a^{\mathrm{c\,BBF}} &\to \boldsymbol{\Sigma}_a(\boldsymbol{x}_b, \mathbf{p}_b, \mathbf{S}_b) \end{aligned}$$

- The values of the coefficients $a_1, \ldots, a_6$, $b_1, \ldots, b_6$ must fulfill the equations

$$E(\mathbf{Y}_a(\boldsymbol{x}_b, \mathbf{p}_b, \mathbf{S}_b), \mathbf{V}_a(\boldsymbol{x}_b, \mathbf{p}_b, \mathbf{S}_b), \boldsymbol{\Sigma}_a(\boldsymbol{x}_b, \mathbf{p}_b, \mathbf{S}_b)) = H(\boldsymbol{x}_a, \mathbf{p}_a, \mathbf{S}_a),$$

$$\mathbf{P}(\mathbf{Y}_a(\boldsymbol{x}_b, \mathbf{p}_b, \mathbf{S}_b), \mathbf{V}_a(\boldsymbol{x}_b, \mathbf{p}_b, \mathbf{S}_b), \boldsymbol{\Sigma}_a(\boldsymbol{x}_b, \mathbf{p}_b, \mathbf{S}_b)) = \sum_a \mathbf{p}_a,$$

$$\mathbf{J}(\mathbf{Y}_a(\boldsymbol{x}_b, \mathbf{p}_b, \mathbf{S}_b), \mathbf{V}_a(\boldsymbol{x}_b, \mathbf{p}_b, \mathbf{S}_b), \boldsymbol{\Sigma}_a(\boldsymbol{x}_b, \mathbf{p}_b, \mathbf{S}_b)) = \sum_a (\mathbf{x}_a \times \mathbf{p}_a + \mathbf{S}_a),$$

$$\mathbf{G}(\mathbf{Y}_a(\boldsymbol{x}_b, \mathbf{p}_b, \mathbf{S}_b), \mathbf{V}_a(\boldsymbol{x}_b, \mathbf{p}_b, \mathbf{S}_b), \boldsymbol{\Sigma}_a(\boldsymbol{x}_b, \mathbf{p}_b, \mathbf{S}_b)) = \mathbf{G}(\boldsymbol{x}_a, \mathbf{p}_a, \mathbf{S}_a).$$

- One obtains a unique set of values for all coefficients $a_1, \ldots, a_6$, $b_1, \ldots, b_6$.

The Transformation Between Orbital Degrees of Freedom (4/4)

Second Method

- One compares the right-hand side of the 2PN-accurate translational equations of motion,

$$\frac{\mathrm{d}\mathbf{v}_a}{\mathrm{d}t} = \mathbf{A}_a(\mathbf{y}_b, \mathbf{v}_b, \mathbf{S}_b^{\mathrm{c}\ \mathrm{BBF}}),$$

  to its direct Hamiltonian recomputation

$$\mathbf{A}_a = \{\mathbf{V}_a, H\} = \{\{\mathbf{Y}_a, H\}, H\}.$$

Bibliography

1. G. Faye, L. Blanchet, and A. Buonanno,
Higher-order spin effects in the dynamics of compact binaries. I. Equations of motion,
Phys. Rev. D **74**, 104033 (2006) [arXiv:gr-qc/0605139].

2. L. Blanchet, A. Buonanno, and G. Faye,
Higher-order spin effects in the dynamics of compact binaries. II. Radiation field,
Phys. Rev. D **74**, 104034 (2006) [arXiv:gr-qc/0605140], Erratum: *ibid.* **75**, 049903(E) (2007).

3. T. Damour, P. Jaranowski, and G. Schäfer,
Hamiltonian of two spinning compact bodies with next-to-leading order gravitational spin-orbit coupling,
Phys. Rev. D **77**, 064032 (2008) [arXiv:0711.1048 [gr-qc]].

4. T. Damour, P. Jaranowski, and G. Schäfer,
Effective one body approach to the dynamics of two spinning black holes with next-to-leading order spin-orbit coupling,
Phys. Rev. D **78**, 024009 (2008) [arXiv:0803.0915 [gr-qc]].