Post-Newtonian General Relativity and Gravitational Waves.
Part II: The ADM Hamiltonian Formalism for 2-Point-Mass Systems

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The "Weimar-Triangle" Collaboration: Post-Newtonian Two-Body Problem, Effective-One-Body Approach

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General Remarks and Summary

- We consider a system of two point masses, i.e. monopolar, pointlike bodies, which interact gravitationally according to general relativity theory. Spin- and tidal-related effects will not be discussed here.
- We model point masses by means of Dirac δ distributions.
- We employ the ADM canonical formalism in $D = d + 1$ spacetime dimensions.
- We work in asymptotically flat $(d+1)$-dimensional spacetime and use asymptotically Minkowskian reference frame with coordinates

$$x^0 = c\,t, \quad \mathbf{x} = (x^1, \dots, x^d).$$

- To solve (perturbatively) equations for the field degrees of freedom, we use time-symmetric (half-retarded half-advanced) Green function for conservative dynamics and retarded one for dissipative dynamics.
- We unambiguously computed conservative Hamiltonians at Newtonian, 1PN, 2PN, 3PN, and 4PN orders and dissipative Hamiltonians at 2.5PN and 3.5PN orders.
- δ-sources lead to ultraviolet (UV) divergences, i.e., divergences at the location of the particles. We control them by means of dimensional regularization (DR).
- For conservative dynamics, near-zone infrared (IR) divergences, linked to nonlocal-in-time tail effects, are analytically regulated using a new (i.e., different from DR-related one) length scale. The result of regularization is ambiguous and the ambiguity is resolved by using a beyond-near-zone information.

Why Dirac Deltas?

♣ Usage of δ-sources considerably simplifies computations.

♣ Effacement principle (Damour 1983): dimensions and internal structure of compact and nonrotating bodies enter their EOM only at the 5PN order.

♣ One can use δs to model source terms for black-hole spacetimes, e.g. the Brill-Lindquist (1963) solution of time-symmetric two-black-hole initial value problem (Jaranowski & Schäfer 1999).

♣ For two-body systems δ-sources together with dimensional regularization give unique conservative EOM up to the 4PN order and made it possible to calculate gravitational-wave luminosities up to the 4.5PN order.

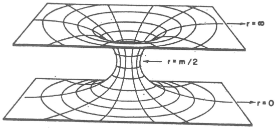

Fig. 1. A two-dimensional analog of the Schwarzschild-Kruskal manifold is shown isometrically imbedded in flat three-space. The figure shows the curvature and topology of the metric

$$ds^2 = (1+m/2r)^4(dr^2+r^2d\theta^2).$$

The sheets at the top and bottom of the funnel continue to infinity and represent the asymptotically flat regions of the manifold ($r\to 0$, $r\to\infty$).

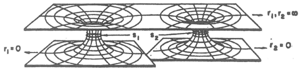

Fig. 2. A two-dimensional analog of the hypersurface of time symmetry of a manifold containing two "throats" is shown isometrically imbedded in flat three-space. The figure illustrates the curvature and topology for a system of two "particles" of equal mass m, and separation large compared to m, described by the metric

$$ds^2 = (1+m/2r_1+m/2r_2)^4 ds_F^2.$$

Dimensional-Regularization Length Scale and Units

- Dimensional regularization introduces a natural length scale ℓ_0, which relates the Newtonian G_N (valid in $d = 3$ space dimensions) and the D-dimensional G_D ($D = d + 1$, valid in d space dimensions) gravitational constants,

$$G_D = G_\mathrm{N}\, \ell_0^{\varepsilon}, \quad \varepsilon := d - 3.$$

- Units: quite often $c = 1$ and $G_D = 1/(16\pi)$.

Notation

Particle labels: $a, b \in \{1, 2\}$,

masses of the particles: m_a,

position vectors of the particles: $\mathbf{x}_a = (x_a^1, \ldots, x_a^d)$,

linear momentum vectors of the particles: $\mathbf{p}_a = (p_{a1}, \ldots, p_{ad})$.

For any d-vectors $\mathbf{v} = (v^1, \ldots, v^d)$ and $\mathbf{w} = (w^1, \ldots, w^d)$:

$$\mathbf{v} \cdot \mathbf{w} := \delta_{ij} v^i w^j, \quad |\mathbf{v}| := \sqrt{\mathbf{v} \cdot \mathbf{v}}.$$

$$\mathbf{r}_a := \mathbf{x} - \mathbf{x}_a, \quad r_a := |\mathbf{r}_a|, \quad \mathbf{n}_a := \mathbf{r}_a / r_a;$$

$$\text{for } a \neq b\text{:} \quad \mathbf{r}_{ab} := \mathbf{x}_a - \mathbf{x}_b, \quad r_{ab} := |\mathbf{r}_{ab}|, \quad \mathbf{n}_{ab} := \mathbf{r}_{ab} / r_{ab}.$$

A $(d+1)$-Splitting of Spacetime Metric $g_{\mu\nu}$

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -(N\,\mathrm{d}t)^2 + \gamma_{ij}(\mathrm{d}x^i + N^i\,\mathrm{d}t)(\mathrm{d}x^j + N^j\,\mathrm{d}t),$$

where N and N^i are respectively *lapse* and *shift* functions,

$$\gamma_{ij} := g_{ij}, \quad N := (-g^{00})^{-1/2}, \quad N^i = \gamma^{ij} N_j \quad \text{with} \quad N_i := g_{0i},$$

here γ^{ij} is the metric inverse to γ_{jk} ($\gamma^{ij}\gamma_{jk} = \delta^i_k$),

$$\gamma := \det(\gamma_{ij});$$

lowering and raising of spatial indices is with γ_{ij}.

Canonical Matter+Field Variables

Canonical matter variables:

$$\left.\begin{aligned} \mathbf{x}_a &= (x_a^1, \ldots, x_a^d), \\ \mathbf{p}_a &= (p_{a1}, \ldots, p_{ad}), \end{aligned}\right. \qquad a = 1, 2.$$

Canonical field variables:

$$\gamma_{ij} := g_{ij},$$

$$\pi^{ij} := \sqrt{\gamma}(K^{ij} - \gamma^{ij}\gamma^{kl}K_{kl}),$$

K_{ij} is the extrinsic curvature of the hypersurface $t = const$.

ADM Hamiltonian

- The full Einstein field equations in D dimensions in an asymptotically flat space-time and in an asymptotically Minkowskian coordinate system are derivable from the Hamiltonian

$$H[\mathbf{x}_a, \mathbf{p}_a, \gamma_{ij}, \pi^{ij}, N, N^i] = \int \mathrm{d}^d x \, (N\mathcal{H} - N^i \mathcal{H}_i) + \oint_{i^0} \mathrm{d}^{d-1} S_i \, \partial_j (\gamma_{ij} - \delta_{ij}\gamma_{kk}),$$

  i^0 denotes spacelike infinity and $\mathrm{d}^{d-1}S_i$ is the $(d-1)$-dimensional out-pointing surface element there.

- The super-Hamiltonian $\mathcal{H}$ and super-momentum $\mathcal{H}_i$ are defined as follows:

$$\mathcal{H}(\mathbf{x}_a, \mathbf{p}_a, \gamma_{ij}, \pi^{ij}) := \sqrt{\gamma} N^2 \left(T^{00} - 2G^{00} \right),$$
$$\mathcal{H}_i(\mathbf{x}_a, \mathbf{p}_a, \gamma_{ij}, \pi^{ij}) := \sqrt{\gamma} N \left(T_i^0 - 2G_i^0 \right).$$

  where $T^{\mu\nu}$ and $G^{\mu\nu}$ denote the energy-momentum and the Einstein tensor, respectively,

Constraint Equations

The lapse and shift functions are Lagrangian multipliers and deliver the Hamiltonian and momentum constraint equations of the Einstein theory,

$$\mathcal{H} = 0, \quad \mathcal{H}_i = 0.$$

2-Point-Mass Energy-Momentum Tensor

- Source terms for the constraint equations are derived from the 2-point-mass energy-momentum tensor

$$T^{\alpha\beta}(x^\mu) := \sum_{a=1}^{2} m_a \int_{-\infty}^{+\infty} \frac{u_a^\alpha u_a^\beta}{\sqrt{-\det(g_{\mu\nu})}} \delta^{d+1}\big(x^\mu - \xi_a^\mu(\tau_a)\big)\, \mathrm{d}\tau_a,$$

  τ_a is the proper time along the world line $x^\mu = \xi_a^\mu(\tau_a)$ of the ath particle, and $u_a^\alpha := \mathrm{d}\xi_a^\alpha / \mathrm{d}\tau_a$.

Constraint Equations for 2-Point-Mass Systems

- The constraint equations:

$$\sqrt{\gamma}\, R - \frac{1}{\sqrt{\gamma}} \left(\gamma_{ik}\, \gamma_{j\ell}\, \pi^{ij}\, \pi^{k\ell} - \frac{(\gamma_{ij}\, \pi^{ij})^2}{d-1} \right) = \sum_{a=1}^{2} \sqrt{\boxed{\gamma_a^{ij}}\, p_{ai} p_{aj} + m_a^2}\, \delta^d(\mathbf{x} - \mathbf{x}_a),$$

$$-2D_j \pi^{ij} = \sum_{a=1}^{2} \boxed{\gamma_a^{ij}}\, p_{aj}\, \delta^d(\mathbf{x} - \mathbf{x}_a),$$

R is the spatial scalar curvature of the hypersurface $t = const$,
D_j is the spatial d-dimensional covariant derivative
(acting on a tensor density of weight one),
$\gamma_a^{ij} := \gamma_{\text{reg}}^{ij}(\mathbf{x}_a)$ is perturbatively unambigously defined and finite
(at least up to the 4PN order).

- The ADM Transverse-Traceless (TT) gauge:

$$\gamma_{ij} = \left(1 + \frac{d-2}{4(d-1)}\phi\right)^{4/(d-2)} \delta_{ij} + h^{\mathrm{TT}}_{ij}, \quad \pi^{ii} = 0,$$

where $h^{\mathrm{TT}}_{ii} = 0$ and $\partial_j h^{\mathrm{TT}}_{ij} = 0$.

- Splitting of the field momentum:

$$\pi^{ij} = \tilde{\pi}^{ij}(V^k) + \pi^{ij}_{\mathrm{TT}},$$
$$\tilde{\pi}^{ij}(V^k) = \partial_i V^j + \partial_j V^i - \frac{2}{d}\,\delta^{ij}\,\partial_k V^k,$$

where $\pi^{ii}_{\mathrm{TT}} = 0$ and $\partial_j \pi^{ij}_{\mathrm{TT}} = 0$.

The super/subscript TT denotes the application of the d-dimensional (spatially nonlocal) TT-projection operator:

$$f^{\mathrm{TT}}_{ij} := \delta^{\mathrm{TT}kl}_{ij} f_{kl},$$

where
$$\delta^{\mathrm{TT}kl}_{ij} := \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) - \frac{1}{d-1}\delta_{ij}\delta_{kl} - \frac{1}{2}(\delta_{ik}\partial_j\partial_l + \delta_{jl}\partial_i\partial_k + \delta_{il}\partial_j\partial_k + \delta_{jk}\partial_i\partial_l)\Delta^{-1} + \frac{1}{d-1}(\delta_{ij}\partial_k\partial_l + \delta_{kl}\partial_i\partial_j)\Delta^{-1} + \frac{d-2}{d-1}\partial_i\partial_j\partial_k\partial_l\Delta^{-2}.$$

Fixing the Gauge: ADMTT Gauge (2/2)

- Asymptotic behavior for $r \to \infty$:

$$\phi \sim \frac{1}{r^{d-2}}, \quad h_{ij}^{\mathrm{TT}} \sim \frac{1}{r^{d-2}}, \quad \tilde{\pi}^{ij} \sim \frac{1}{r^{d-1}}, \quad \pi_{\mathrm{TT}}^{ij} \sim \frac{1}{r^{d-1}}.$$

- After PN expansion of the retardations in the field functions one gets new functions which behave badly for $r \to \infty$:

$$\begin{aligned} h_{ij}^{\mathrm{TT}}(t, \mathbf{n}r) &= \frac{h_{ij}(t-r, \mathbf{n})}{r^{d-2}} + \mathcal{O}(\frac{1}{r^{d-1}}) \\ &= \frac{h_{ij}(t, \mathbf{n})}{r^{d-2}} - \dot{h}_{ij}(t, \mathbf{n}) r^{3-d} + \frac{1}{2}\ddot{h}_{ij}(t, \mathbf{n}) r^{4-d} + \cdots + \mathcal{O}(\frac{1}{r^{d-1}}). \end{aligned}$$

This is the source of infrared divergences.

A Perturbative Solving of the Constraints (1/3)

- ϕ and V^i are expressed in terms of $(\mathbf{x}_a, \mathbf{p}_a, h_{ij}^{\mathrm{TT}}, \pi_{\mathrm{TT}}^{ij})$ by a perturbative solving of the constraint equations—this is done by the PN expansion of ϕ and V^i, which is slow-motion and weak-field approximation, so we assume that

$$\frac{v^2}{c^2} \sim \frac{G_D m}{c^2 r^{d-2}} \ll 1.$$

  Working with $c = 1$ units we thus have $v = \mathcal{O}(c^{-1})$ and $m = \mathcal{O}(c^{-2})$.

- One take into account that

$$m_a \sim \mathcal{O}(c^{-2}), \quad \mathbf{p}_a \sim \mathcal{O}(c^{-3}), \quad \phi \sim \mathcal{O}(c^{-2}),$$
$$h_{ij}^{\mathrm{TT}} \sim \mathcal{O}(c^{-4}), \quad \tilde{\pi}^{ij} \sim V^i \sim \mathcal{O}(c^{-3}), \quad \pi_{\mathrm{TT}}^{ij} \sim \mathcal{O}(c^{-5}).$$

- The PN expansion of ϕ and V^i read (the numbers in parentheses denote the formal order in $1/c$):

$$\phi = \phi_{(2)} + \phi_{(4)} + \cdots, \quad V^i = V^i_{(3)} + V^i_{(5)} + \cdots.$$

- The constraints yield a system of elliptic equations for ϕ and V^i, which has the structure (h_{ij}^{TT} and π_{TT}^{ij} enter the ellipsis)

$$\Delta\phi = -\sum_a m_a(1 + \cdots)\,\delta^d(\mathbf{x} - \mathbf{x}_a) + \cdots,$$
$$\Delta V^i + \left(1 - \frac{2}{d}\right)\partial_i\partial_j V^j = -\frac{1}{2}\sum_a (p_{ai} + \cdots)\,\delta^d(\mathbf{x} - \mathbf{x}_a) + \cdots.$$

A Perturbative Solving of the Constraints (2/3)

- The 3PN-accurate conservative matter *Hamiltonian density* can be expressed in terms of the six functions: $\phi_{(2)}$, $S_{(4)1}$, $S_{(4)2}$, $S_{(4)ij}$, $V^i_{(3)}$, $h^{\mathrm{TT}}_{(4)ij}$.
- They satisfy the equations $\big(\delta_a \equiv \delta^d(\mathbf{x}-\mathbf{x}_a)\big)$:

$$\Delta\,\phi_{(2)} = -\sum_a m_a\,\delta_a,$$

$$\Delta\,V^i_{(3)} + \left(1-\frac{2}{d}\right)\partial_{ij}V^j_{(3)} = -\frac{1}{2}\sum_a p_{ai}\,\delta_a,$$

$$\Delta S_{(4)1} = \sum_a \frac{\mathbf{p}_a^2}{m_a}\delta_a, \quad \Delta S_{(4)2} = \phi_{(2)}\sum_a m_a\delta_a, \quad \Delta S_{(4)ij} = \sum_a \frac{p_{ai}p_{aj}}{m_a}\delta_a,$$

$$\Delta h^{\mathrm{TT}}_{(4)ij} = \left(-\sum_a \frac{p_{ai}p_{aj}}{m_a}\delta_a - \frac{d-2}{2(d-1)}\partial_i\phi_{(2)}\partial_j\phi_{(2)}\right)^{\mathrm{TT}}.$$

A Perturbative Solving of the Constraints (3/3)

- Using the relations:

$$\Delta^{-1}\delta_a = -\kappa\, r_a^{2-d} \quad \left(\kappa := \frac{\Gamma\left(d/2-1\right)}{4\pi^{d/2}}\right),$$

$$\Delta^{-1} r_a^{\lambda} = \frac{r_a^{\lambda+2}}{(\lambda+2)(\lambda+d)},$$

one can find in d dimensions $\phi_{(2)}$, $\phi_{(4)}$, $V^i_{(3)}$, $S_{(4)}$, $S_{(4)ij}$, and the quadratic in momenta part of $h^{\mathrm{TT}}_{(4)ij}$.

- E.g., $\phi_{(2)} = -\sum_a m_a\, \Delta^{-1}\delta_a = \kappa \sum_a m_a r_a^{2-d}$.

Reduced Matter+Field ADM Hamiltonian

- If the constraint equations and the gauge conditions are both satisfied, the total matter+field ADM Hamiltonian can be written in its reduced form:

$$H_{\rm red}\left[\mathbf{x}_a, \mathbf{p}_a, h_{ij}^{\rm TT}, \pi^{ij}_{\rm TT}\right] = -\sum_{n=2}^{\infty} \int \mathrm{d}^d\mathbf{x}\, \Delta\phi_{(n)}\left[\mathbf{x}_a, \mathbf{p}_a, h_{ij}^{\rm TT}, \pi^{ij}_{\rm TT}\right].$$

- The equations of motion for the particles:

$$\dot{\mathbf{p}}_a = -\frac{\delta H_{\rm red}}{\delta \mathbf{x}_a}, \quad \dot{\mathbf{x}}_a = \frac{\delta H_{\rm red}}{\delta \mathbf{p}_a} \quad (a = 1, 2).$$

  Evolution equations for the field degrees of freedom:

$$\frac{\partial}{\partial t} h_{ij}^{\rm TT} = \delta_{ij}^{{\rm TT}kl} \frac{\delta H_{\rm red}}{\delta \pi^{kl}_{\rm TT}}, \quad \frac{\partial}{\partial t} \pi^{ij}_{\rm TT} = -\delta_{kl}^{{\rm TT}ij} \frac{\delta H_{\rm red}}{\delta h_{kl}^{\rm TT}}.$$

- There is no involvement of lapse and shift functions in the equations of motion and in the field equations for the independent degrees of freedom.

Field Equations (1/2)

- For computing the 4PN-accurate reduced Hamiltonian we need to use field equations which follow from the 3PN-accurate part of the Hamiltonian:

$$H^{\rm red}_{\le 3\rm PN}\big[\mathbf{x}_a, \mathbf{p}_a, h^{\rm TT}_{ij}, \pi^{ij}_{\rm TT}\big] = \int \mathrm{d}^d\mathbf{x}\, h_{\le 3\rm PN}\big(\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a, h^{\rm TT}_{ij}, \pi^{ij}_{\rm TT}\big),$$

where

$$h_{\le 3\rm PN}\big(\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a, h^{\rm TT}_{ij}, \pi^{ij}_{\rm TT}\big) = \sum_a m_a\delta_a + h_{(4)}\big(\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a\big),$$
$$+ h_{(6)}\big(\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a\big) + h_{(8)}\big(\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a, h^{\rm TT}_{ij}\big) + h_{(10)}\big(\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a, h^{\rm TT}_{ij}, \pi^{ij}_{\rm TT}\big).$$

- For this Hamiltonian the field equations take the form

$$\dot h^{\rm TT}_{ij} = \delta^{{\rm TT}kl}_{ij}\,\frac{\partial h_{\le 3\rm PN}}{\partial \pi^{kl}_{\rm TT}} + \mathcal{O}(c^{-7}),$$
$$\dot \pi^{ij}_{\rm TT} = -\delta^{{\rm TT}ij}_{kl}\left\{\frac{\partial h_{\le 3\rm PN}}{\partial h^{\rm TT}_{kl}} - \left(\frac{\partial h_{\le 3\rm PN}}{\partial h^{\rm TT}_{kl,m}}\right)_{,m} + \left(\frac{\partial h_{\le 3\rm PN}}{\partial h^{\rm TT}_{kl,mn}}\right)_{,mn}\right\} + \mathcal{O}(c^{-8}).$$

Field Equations (2/2)

- More explicitly,

$$\dot{h}_{ij}^{\mathrm{TT}} = \delta_{ij}^{\mathrm{TT}kl}\left\{2\pi_{\mathrm{TT}}^{kl} - \frac{2(d-2)}{d-1}\phi_{(2)}\tilde{\pi}_{(3)}^{kl}\right\} + \mathcal{O}(c^{-7}),$$

$$\dot{\pi}_{\mathrm{TT}}^{ij} = -\delta_{ij}^{\mathrm{TT}kl}\left\{\frac{1}{2}S_{(4)kl} - \frac{1}{2}\Delta h_{kl}^{\mathrm{TT}} + B_{(6)kl} + \frac{1}{2(d-1)}\left(\phi_{(2)}\Delta h_{kl}^{\mathrm{TT}} + \Delta\left(\phi_{(2)}h_{kl}^{\mathrm{TT}}\right)\right)\right\} + \mathcal{O}(c^{-8}).$$

- By combining these two equations one gets the equation for h_{ij}^{TT},

$$\Box h_{ij}^{\mathrm{TT}} = S_{ij}^{\mathrm{TT}}, \quad \Box := -\partial_t^2 + \Delta,$$

  where the source term is

$$S_{ij}^{\mathrm{TT}} = \delta_{ij}^{\mathrm{TT}kl}\left\{S_{(4)kl} + 2B_{(6)kl} + \frac{2(d-2)}{d-1}\partial_t\left(\phi_{(2)}\tilde{\pi}_{(3)}^{kl}\right) + \frac{1}{d-1}\left(\phi_{(2)}\Delta h_{kl}^{\mathrm{TT}} + \Delta\left(\phi_{(2)}h_{kl}^{\mathrm{TT}}\right)\right)\right\} + \mathcal{O}(c^{-8}).$$

- After solving field equation for h_{ij}^{TT} one can obtain π_{TT}^{ij}:

$$\pi_{\mathrm{TT}}^{ij} = \frac{1}{2}\dot{h}_{ij}^{\mathrm{TT}} + \frac{d-2}{d-1}\delta_{ij}^{\mathrm{TT}kl}\left(\phi_{(2)}\tilde{\pi}_{(3)}^{kl}\right) + \mathcal{O}(c^{-7}).$$

Conservative Matter Hamiltonian

From this point on, we limit ourselves to the conservative dynamics.

- We further reduce of the Hamiltonian by performing the Legendre transformation with respect to the field variables. This leads to the Routhian,

$$R\left[\mathbf{x}_a, \mathbf{p}_a, h_{ij}^{\mathrm{TT}}, \dot{h}_{ij}^{\mathrm{TT}}\right] := H_{\text{red}}\left[\mathbf{x}_a, \mathbf{p}_a, h_{ij}^{\mathrm{TT}}, \dot{h}_{ij}^{\mathrm{TT}}\right] - \int \mathrm{d}^d\mathbf{x}\, \pi_{\mathrm{TT}}^{ij} \dot{h}_{ij}^{\mathrm{TT}}.$$

- Elimination of the field variables h_{ij}^{TT}, $\dot{h}_{ij}^{\mathrm{TT}}$: they are "integrating out", i.e., replaced by time-symmetric solutions (as a functional of the particle variables) of their field equations,

$$H_{\text{cons}}[\mathbf{x}_a, \mathbf{p}_a] := R\left[\mathbf{x}_a, \mathbf{p}_a, h_{\text{sym}\,ij}^{\mathrm{TT}}(\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a), \dot{h}_{\text{sym}\,ij}^{\mathrm{TT}}(\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a)\right],$$

  where time derivatives of $\mathbf{x}_a$ and $\mathbf{p}_a$ are eliminated through the use of lower-order equations of motion.

- The reduced action for the particles (in Hamiltonian form) is

$$S = \sum_a \int \mathbf{p}_a \cdot \mathrm{d}\mathbf{x}_a - \int \mathrm{d}t\, H_{\text{cons}}[\mathbf{x}_a, \mathbf{p}_a].$$

Conservative Matter Hamiltonian: Near-Zone Contribution (1/2)

- Equations for the field degrees of freedom can be combined to get

$$\Box h^{\mathrm{TT}}_{ij} = S^{\mathrm{TT}}_{ij}, \quad \Box := -c^{-2}\partial_t^2 + \Delta,$$

$$S_{ij}(\mathbf{x},t) = S_{ij}\big(\mathbf{x}-\mathbf{x}_a(t), \mathbf{p}_a(t), h^{\mathrm{TT}}_{ij}(\mathbf{x},t), \pi^{ij}_{\mathrm{TT}}(\mathbf{x},t)\big).$$

- Time-symmetric and near-zone solution of the field equation,

$$h^{\mathrm{TT\,loc}}_{\mathrm{sym}\,ij} = \left(\Box^{-1}_{\mathrm{sym}} S_{ij}\right)^{\mathrm{TT}} = \frac{1}{2}\left((\Box^{-1}_{\mathrm{adv}} + \Box^{-1}_{\mathrm{ret}})S_{ij}\right)^{\mathrm{TT}}$$

$$= \left((\Delta^{-1} + c^{-2}\Delta^{-2}\partial_t^2 + c^{-4}\Delta^{-3}\partial_t^4 + \cdots)S_{ij}\right)^{\mathrm{TT}}.$$

- After making the PN expansion of the source terms, $S_{ij} = S_{(4)ij} + S_{(6)ij} + \cdots$, one gets $h^{\mathrm{TT\,loc}}_{\mathrm{sym}\,ij}(\mathbf{x},t) = h^{\mathrm{TT}}_{(4)ij}(\mathbf{x},t) + h^{\mathrm{TT}}_{(6)ij}(\mathbf{x},t) + \cdots$, $\Delta h^{\mathrm{TT}}_{(4)ij} = S^{\mathrm{TT}}_{(4)ij}$, $\Delta h^{\mathrm{TT}}_{(6)ij} = S^{\mathrm{TT}}_{(6)ij} + \ddot{h}^{\mathrm{TT}}_{(4)ij}$. The functions $h^{\mathrm{TT}}_{(4)ij}$ and $h^{\mathrm{TT}}_{(6)ij}$ are enough to compute 4PN-accurate conservative Hamiltonian.

- Slow decay of $h^{\mathrm{TT}}_{(4)ij}(\mathbf{x},t)$ (like $1/r$ in $d=3$) and divergence of $h^{\mathrm{TT}}_{(6)ij}(\mathbf{x},t)$ (like r in $d=3$) for $r := |\mathbf{x}| \to \infty$ leads to infrared (IR) divergences. To regularize them one needs to introduce a new length scale s_{IR}.

Conservative Matter Hamiltonian: Near-Zone Contribution (2/2)

- After replacing $h^{\mathrm{TT}}_{\mathrm{sym}\,ij}$ by $h^{\mathrm{TT}\,\mathrm{loc}}_{\mathrm{sym}\,ij}$ one gets the near-zone conservative Hamiltonian, which is local in time:

$$H_{\text{near-zone}}(\mathbf{x}_a, \mathbf{p}_a) := R\left[\mathbf{x}_a, \mathbf{p}_a, h^{\mathrm{TT}\,\mathrm{loc}}_{\mathrm{sym}\,ij}(\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a), \dot{h}^{\mathrm{TT}\,\mathrm{loc}}_{\mathrm{sym}\,ij}(\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a)\right].$$

- The Hamiltonian $H_{\text{near-zone}}$ develops both UV and IR divergences:

$$\mathrm{Reg}\left\{H_{\text{near-zone}}(\mathbf{x}_a, \mathbf{p}_a)\right\} = \mathrm{Reg}_{\mathrm{UV}}\left\{H^{\mathrm{IR\ conv}}_{\text{near-zone}}(\mathbf{x}_a, \mathbf{p}_a)\right\} + \mathrm{Reg}^{s_{\mathrm{IR}}}_{\mathrm{IR}}\left\{H^{\mathrm{IR\ div}}_{\text{near-zone}}(\mathbf{x}_a, \mathbf{p}_a)\right\}.$$

  The result of the IR regularization depends on the scale s_{IR}.

Conservative Matter Hamiltonian: Tail Contribution (1/2)

- Work of Blanchet & Damour (1988):
  - — starting at the 4PN level it is impossible to express (in any gauge) the near-zone metric as a functional of the instantaneous state of the source: 4PN metric is the sum of an instantaneous functional of the source variables and of a nonlocal-in-time tail contribution;
  - — to compute the near-zone effect of tail-transported correlations a technique of matching between the exterior zone $r \gg r_{12}$ and the near-zone $r \ll \lambda/(2\pi)$ was employed;
  - — its result depends on an arbitrary length scale s_{tail} which plays the role of an intermediate scale between the scale of the system r_{12} and the reduced wavelength $\lambda/(2\pi)$,

$$r_{12} \ll s_{\text{tail}} \ll \lambda/(2\pi).$$

Conservative Matter Hamiltonian: Tail Contribution (2/2)

- The time-symmetric part of the 4PN tail metric (Blanchet & Damour 1988) contributes to the two-body EOM through a nonlocal-in-time Hamiltonian (Damour, Jaranowski, & Schäfer 2014) in $d = 3$ dimensions:

$$\mathrm{Reg}^{s_{\mathrm{tail}}}\left\{H_{\mathrm{4PN}}^{\mathrm{tail\ sym}}[\mathbf{x}_a, \mathbf{p}_a]\right\} = -\frac{1}{5}\,\frac{G^2 M}{c^8}\,\dddot{I}_{ij}\,\mathrm{Pf}_{2s_{\mathrm{tail}}/c}\int_{-\infty}^{+\infty}\frac{\mathrm{d}v}{|v|}\,\dddot{I}_{ij}(t+v),$$

where Pf_T denotes a Hadamard partie finie with time scale $T := 2s_{\mathrm{tail}}/c$,

$$\mathrm{Pf}_T\int_{-\infty}^{+\infty}\frac{\mathrm{d}v}{|v|}\,g(v) := \int_{-T}^{T}\frac{\mathrm{d}v}{|v|}\,\big(g(v)+g(-v)-2g(0)\big) + \int_{-\infty}^{-T}\frac{\mathrm{d}v}{|v|}\,g(v) + \int_{T}^{+\infty}\frac{\mathrm{d}v}{v}\,g(v),$$

and I_{ij} is the Newtonian quadrupole moment of the binary system:

$$I_{ij} := \sum_a m_a\left(x_a^i\,x_a^j - \frac{1}{3}\,\delta^{ij}\,\mathbf{x}_a^2\right).$$

Conservative Matter Hamiltonian

- One identifies the IR-related and the tail-related regularization length scales:

$$s_{\mathrm{IR}} = s_{\mathrm{tail}} \equiv s.$$

- The total conservative Hamiltonian is the sum of the near-zone contribution and the time-symmetric part of the tail contribution:

$$H_{\mathrm{cons}}[\mathbf{x}_a, \mathbf{p}_a] = H_{\text{near-zone}}(\mathbf{x}_a, \mathbf{p}_a) + H_{\text{tail sym}}[\mathbf{x}_a, \mathbf{p}_a].$$

3.5PN-Accurate Dissipative Matter Hamiltonian (1/2)

- More refined treatment can be found in Section 3 of 2018 Schäfer/Jaranowski *Living Reviews in Relativity* article.
- Now we use the near-zone expansion of the retarded solutions of the field equations,

$$h_{ij}^{\mathrm{TT}} = h_{(4)ij}^{\mathrm{TT}} + h_{(5)ij}^{\mathrm{TT}} + h_{(6)ij}^{\mathrm{TT}} + h_{(7)ij}^{\mathrm{TT}} + \mathcal{O}(c^{-8}),$$
$$\pi_{\mathrm{TT}}^{ij} = \pi_{\mathrm{TT}}^{(5)ij} + \pi_{\mathrm{TT}}^{(6)ij} + \mathcal{O}(c^{-7}).$$

- The split of the total reduced Hamiltonian:

$$H_{\le 3.5\mathrm{PN}}^{\mathrm{red}}\left[\mathbf{x}_a, \mathbf{p}_a, h_{ij}^{\mathrm{TT}}, \pi_{\mathrm{TT}}^{ij}\right] = H_{\le 3.5\mathrm{PN}}^{\mathrm{mat}}\left(\mathbf{x}_a, \mathbf{p}_a\right) + H_{\le 3.5\mathrm{PN}}^{\mathrm{int}}\left[\mathbf{x}_a, \mathbf{p}_a, h_{ij}^{\mathrm{TT}}, \pi_{\mathrm{TT}}^{ij}\right] + H_{\le 3.5\mathrm{PN}}^{\mathrm{field}}\left[h_{ij}^{\mathrm{TT}}, \pi_{\mathrm{TT}}^{ij}\right].$$

- $\widetilde{H}_{\le 3.5\mathrm{PN}}$ is the Hamiltonian which coincides with the Hamiltonian $H_{\le 3.5\mathrm{PN}}$ after dropping its field part,

$$\widetilde{H}_{\le 3.5\mathrm{PN}} := H_{\le 3.5\mathrm{PN}}^{\mathrm{mat}} + H_{\le 3.5\mathrm{PN}}^{\mathrm{int}}.$$

3.5PN-Accurate Dissipative Matter Hamiltonian (2/2)

- The Hamiltonian $\widetilde{H}_{\leq 3.5\mathrm{PN}}$ can be decomposed into conservative and dissipative parts

$$\widetilde{H}_{\leq 3.5\mathrm{PN}}\left(\mathbf{x}_a, \mathbf{p}_a, t\right) = H^{\mathrm{con}}_{\leq 3\mathrm{PN}}\left(\mathbf{x}_a, \mathbf{p}_a\right) + H^{\mathrm{diss}}_{\leq 3.5\mathrm{PN}}\left(\mathbf{x}_a, \mathbf{p}_a, t\right),$$

where

$$H^{\mathrm{con}}_{\leq 3\mathrm{PN}} := H^{\mathrm{mat}}_{\mathrm{N}} + H^{\mathrm{mat}}_{1\mathrm{PN}} + \left(H^{\mathrm{mat}}_{2\mathrm{PN}} + H^{\mathrm{int}}_{2\mathrm{PN}}\right) + \left(H^{\mathrm{mat}}_{3\mathrm{PN}} + H^{\mathrm{int}}_{3\mathrm{PN}}\right),$$

$$H^{\mathrm{diss}}_{\leq 3.5\mathrm{PN}}\left(\mathbf{x}_a, \mathbf{p}_a, t\right) := H^{\mathrm{int}}_{2.5\mathrm{PN}}\left[\mathbf{x}_a, \mathbf{p}_a, h^{\mathrm{TT}}_{ij}(t), \pi^{ij}_{\mathrm{TT}}(t)\right] + H^{\mathrm{int}}_{3.5\mathrm{PN}}\left[\mathbf{x}_a, \mathbf{p}_a, h^{\mathrm{TT}}_{ij}(t), \pi^{ij}_{\mathrm{TT}}(t)\right].$$

3.5PN-Accurate Gravitational-Wave Luminosity

- The total time derivative of $\widetilde{H}_{\leq 3.5\mathrm{PN}}$ is equal to its partial time derivative, and because only the dissipative part of $\widetilde{H}_{\leq 3.5\mathrm{PN}}$ depends explicitly on time we get

$$\frac{d}{dt}\widetilde{H}_{\leq 3.5\mathrm{PN}}\left(\mathbf{x}_a, \mathbf{p}_a, t\right) = \frac{\partial}{\partial t}\widetilde{H}_{\leq 3.5\mathrm{PN}}\left(\mathbf{x}_a, \mathbf{p}_a, t\right) = \frac{\partial}{\partial t}H^{\mathrm{diss}}_{\leq 3.5\mathrm{PN}}\left(\mathbf{x}_a, \mathbf{p}_a, t\right).$$

- The instantaneous energy loss of the matter system due to the gravitational wave emission is defined as

$$\mathcal{L}^{\mathsf{inst}}_{\leq 3.5\mathrm{PN}}(t) := -\frac{\partial}{\partial t}H^{\mathrm{diss}}_{\leq 3.5\mathrm{PN}}\left(\mathbf{x}_a, \mathbf{p}_a, t\right).$$

- This formula was applied to derive, at the leading-order (the quadrupole formula) and the next-to-leading-order, gravitational-wave luminosity $\mathcal{L}$ of the two-body system in quasi-elliptical motion:

$$\mathcal{L}_{\leq 3.5\mathrm{PN}} = \left\langle \mathcal{L}^{\mathsf{inst}}_{\leq 3.5\mathrm{PN}}(t) \right\rangle = -\left\langle \frac{\partial}{\partial t}H^{\mathrm{diss}}_{\leq 3.5\mathrm{PN}}(\mathbf{x}_a, \mathbf{p}_a, t) \right\rangle,$$

  where $\langle \cdots \rangle$ denotes time averaging over one period of the motion.
  Also the leading-order spin-orbit and spin(1)-spin(2) dissipative Hamiltonians were derived.

- This is a direct derivation of the leading-order/next-to-leading-order gravitational-wave luminosity.

- The leading-order dissipative Hamiltonian

$$H^{\mathrm{diss}}_{2.5\mathrm{PN}}(\mathbf{x}_a, \mathbf{p}_a, t) = \frac{1}{2}\int \mathrm{d}^3\mathbf{x}\, S_{(4)ij}(\mathbf{x}-\mathbf{x}_a, \mathbf{p}_a) h^{\mathrm{TT}}_{(5)ij}(t),$$

depends on function $h^{\mathrm{TT}}_{(5)ij}$,

$$h^{\mathrm{TT}}_{(5)ij}(t) = \frac{1}{4\pi}\partial_t \delta^{\mathrm{TT}kl}_{ij} \int \mathrm{d}^3\mathbf{x}\, S_{(4)kl}(\mathbf{x}-\mathbf{x}_a(t), \mathbf{p}_a(t)).$$

- Taking into account that

$$\delta^{\mathrm{TT}kl}_{ij} T_{kl}(t) = \frac{2}{5} T^{\mathrm{STF}}_{ij}(t), \quad T^{\mathrm{STF}}_{ij} := \frac{1}{2}\left(T_{ij}+T_{ji}\right) - \frac{1}{3}\delta_{ij} T_{kk},$$

one gets

$$h^{\mathrm{TT}}_{(5)ij}(t) = \frac{d}{dt}\chi_{(4)ij}\left(\boldsymbol{x}_a(t), \boldsymbol{p}_a(t)\right),$$

where

$$\chi_{(4)ij}\left(\boldsymbol{x}_a, \boldsymbol{p}_a\right) := \frac{1}{60\pi}\Bigg\{\sum_a \frac{2}{m_a}\left(\boldsymbol{p}_a^2\delta_{ij} - 3p_{ai}p_{aj}\right) + \frac{1}{16\pi}\sum_a\sum_{b\neq a}\frac{m_a m_b}{r_{ab}}\left(3n^i_{ab}n^j_{ab} - \delta_{ij}\right)\Bigg\}.$$

Leading-Order Gravitational-Wave Luminosity (2/2)

- Finally, the leading-order dissipative Hamiltonian reads

$$H^{\mathrm{diss}}_{2.5\mathrm{PN}}\left(\mathbf{x}_a, \mathbf{p}_a, t\right) = 5\pi\, \dot{\chi}_{(4)ij}(t)\, \chi_{(4)ij}\left(\boldsymbol{x}_a, \boldsymbol{p}_a\right),$$

so the leading-order GW luminosity reads

$$\begin{aligned}
\mathcal{L}_{2.5\mathrm{PN}} &= -\left\langle \frac{\partial}{\partial t} H^{\mathrm{diss}}_{2.5\mathrm{PN}}\left(\mathbf{x}_a, \mathbf{p}_a, t\right) \right\rangle \\
&= -5\pi \left\langle \ddot{\chi}_{(4)ij}(t)\, \chi_{(4)ij}\left(\boldsymbol{x}_a, \boldsymbol{p}_a\right) \right\rangle \\
&= -5\pi \left\langle \frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{\chi}_{(4)ij}\chi_{(4)ij}\right) - \left(\dot{\chi}_{(4)ij}\right)^2 \right\rangle \\
&= +5\pi \left\langle \left(\dot{\chi}_{(4)ij}\right)^2 \right\rangle .
\end{aligned}$$

4PN-Accurate Conservative Hamiltonian: UV/IR Divergences

$$H^{\text{cons}}_{\leq 4\text{PN}}[\mathbf{x}_a, \mathbf{p}_a] = H_{\text{N}}(\mathbf{x}_a, \mathbf{p}_a) + H_{1\text{PN}}(\mathbf{x}_a, \mathbf{p}_a) + H_{2\text{PN}}(\mathbf{x}_a, \mathbf{p}_a) + H_{3\text{PN}}(\mathbf{x}_a, \mathbf{p}_a) + H_{4\text{PN}}[\mathbf{x}_a, \mathbf{p}_a];$$

$$H_{\text{N}}(\mathbf{x}_a, \mathbf{p}_a) = \text{Reg}_{\text{UV}}\left\{H^{\text{near-zone}}_{\text{N}}(\mathbf{x}_a, \mathbf{p}_a)\right\},$$

$$H_{1\text{PN}}(\mathbf{x}_a, \mathbf{p}_a) = \text{Reg}_{\text{UV}}\left\{H^{\text{near-zone}}_{1\text{PN}}(\mathbf{x}_a, \mathbf{p}_a)\right\},$$

$$H_{2\text{PN}}(\mathbf{x}_a, \mathbf{p}_a) = \text{Reg}_{\text{UV}}\left\{H^{\text{near-zone}}_{2\text{PN}}(\mathbf{x}_a, \mathbf{p}_a)\right\},$$

$$H_{3\text{PN}}(\mathbf{x}_a, \mathbf{p}_a) = \text{Reg}_{\text{UV}}\left\{H^{\text{near-zone}}_{3\text{PN}}(\mathbf{x}_a, \mathbf{p}_a)\right\},$$

$$H_{4\text{PN}}[\mathbf{x}_a, \mathbf{p}_a] = \text{Reg}_{\text{UV}}\left\{H^{\text{near-zone}}_{4\text{PN, IR conv}}(\mathbf{x}_a, \mathbf{p}_a)\right\} + \text{Reg}^{s}_{\text{IR}}\left\{H^{\text{near-zone}}_{4\text{PN, IR div}}(\mathbf{x}_a, \mathbf{p}_a)\right\} + \text{Reg}^{s}\left\{H^{\text{tail sym}}_{4\text{PN}}[\mathbf{x}_a, \mathbf{p}_a]\right\}.$$

All regularization procedures used are described in detail in Appendix A of 2015 Jaranowski/Schäfer PRD article.

Structure of the Hamiltonian Density: Contact and Field-Like Terms

- The near-zone Hamiltonian can be written as

$$H = \int \mathcal{H}(\mathbf{x})\, \mathrm{d}^d\mathbf{x}\,, \quad \mathcal{H}(\mathbf{x}) = \mathcal{H}^{(\mathcal{D})}_{\text{contact}}(\mathbf{x}) + \mathcal{H}^{(\mathcal{D})}_{\text{field}}(\mathbf{x}) + \partial_i \mathcal{D}^i(\mathbf{x}),$$

  $\partial_i \mathcal{D}^i$ gives no contribution to the H.
  By changing $\mathcal{D}^i$, both $\mathcal{H}^{(\mathcal{D})}_{\text{contact}}$ and $\mathcal{H}^{(\mathcal{D})}_{\text{field}}$ change,
  but H should should remain unchanged.

- One can also shift time derivatives,

$$\int \dot{A}B\, \mathrm{d}^d\mathbf{x} = -\int A\dot{B}\, \mathrm{d}^d\mathbf{x} + \frac{\mathrm{d}}{\mathrm{d}t}\int AB\, \mathrm{d}^d\mathbf{x}.$$

  Dropping the total time derivative is equivalent to performing a canonical transformation.

Structure of Contact and Field-Like Terms

- Structure of contact $\mathcal{H}^{(\mathcal{D})}_{\text{contact}}$ terms:

$$\mathcal{H}^{(\mathcal{D})}_{\text{contact}} = S_1(\mathbf{x})\,\delta^d(\mathbf{x}-\mathbf{x}_1) + (1 \leftrightarrow 2).$$

- Structure of field-like $\mathcal{H}^{(\mathcal{D})}_{\text{field}}$ terms in $d = 3$ dimensions (at least up to 4PN order):

$$\mathcal{H}^{(\mathcal{D})}_{\text{field}} = \sum_{\boldsymbol{\ell}} c_{\boldsymbol{\ell}}\,(\mathbf{n}_1\cdot\mathbf{p}_1)^{\ell_1}\,(\mathbf{n}_2\cdot\mathbf{p}_1)^{\ell_2}\,(\mathbf{n}_1\cdot\mathbf{p}_2)^{\ell_3}\,(\mathbf{n}_2\cdot\mathbf{p}_2)^{\ell_4} \times r_1^{\ell_5}\,r_2^{\ell_6}(r_1+r_2+r_{12})^{\ell_7};$$

using prolate spheroidal coordinates one can reduce field-like integrands to

$$c_{\boldsymbol{\ell}}\,r_1^{\ell_1}\,r_2^{\ell_2}(r_1+r_2+r_{12})^{\ell_3}.$$

Looking for the Correct 3-Dimensional Regularization (1/4)

- "Good" δ-functions of Infeld and Plebański (1954–60); they satisfy, besides having the properties of ordinary Dirac δ-functions, the condition

$$\frac{1}{|\mathbf{x}-\mathbf{x}_a|^k}\delta^3(\mathbf{x}-\mathbf{x}_a)=0, \quad k=1,\dots,p \quad \text{(for some positive integer } p\text{)}.$$

- A natural generalization of the concept of "good" δ-functions is "partie finie" value of function at its singular point $\mathbf{x}_0$ (here $m_{\max}$ is some nonnegative integer):

$$f(\mathbf{x}_a+r_a\mathbf{n}_a)=\sum_{m=-m_{\max}}^{\infty} a_m(\mathbf{n}_a)r_a^m, \quad f_{\mathrm{reg}}(\mathbf{x}_a):=\frac{1}{4\pi}\int \mathrm{d}\Omega\, a_0(\mathbf{n}_a).$$

- All contact integrals are evaluated as

$$\left\{\int \mathrm{d}^3\mathbf{x}\, f(\mathbf{x})\delta^3(\mathbf{x}-\mathbf{x}_a)\right\}_{\mathrm{reg}} := f_{\mathrm{reg}}(\mathbf{x}_a).$$

Looking for the Correct 3-Dimensional Regularization (2/4)

- Infeld and Plebański assumed that the "tweedling of products" is always satisfied:

$$(f_1 f_2)_{\rm reg}(\mathbf{x}_a) = f_{1\rm reg}(\mathbf{x}_a) f_{2\rm reg}(\mathbf{x}_a),$$

  but this is generally wrong for arbitrary singular functions f_1 and f_2.
  Problems with fulfilling this property begin at the 3PN order.

- It is natural to demand that

$$S(\mathbf{x})\,\delta^3(\mathbf{x}-\mathbf{x}_a) = S_{\rm reg}(\mathbf{x}_a)\,\delta^3(\mathbf{x}-\mathbf{x}_a),$$

  then, taking another singular function T, one gets

$$T(\mathbf{x})\,S(\mathbf{x})\,\delta^3(\mathbf{x}-\mathbf{x}_a) = T(\mathbf{x})\,S_{\rm reg}(\mathbf{x}_a)\,\delta^3(\mathbf{x}-\mathbf{x}_a).$$

  This implies $\quad (TS)_{\rm reg}(\mathbf{x}_a) = T_{\rm reg}(\mathbf{x}_a)\,S_{\rm reg}(\mathbf{x}_a).$

- Another consequence of employing δ-sources is necessity to differentiate singular and homogeneous functions using a distributional derivative.
- Let f be a function defined in a neighbourhood of the origin of $\mathbb{R}^3$, it is said to be a positively homogeneous function of degree λ, if for any number $a > 0$

$$f(a\,\mathbf{x}) = a^{\lambda}\, f(\mathbf{x}).$$

Let $k := -\lambda - 2$. If λ is an integer and if $\lambda \leq -2$ (i.e., k is a nonnegative integer), then within the standard distribution theory one derives the formula

$$\bar{\partial}_i f(\mathbf{x}) = \partial_i f(\mathbf{x}) + \frac{(-1)^k}{k!} \frac{\partial^k \delta^3(\mathbf{x})}{\partial x^{i_1} \cdots \partial x^{i_k}} \times \oint_{\Sigma} \mathrm{d}\sigma_i\, f(\mathbf{x}')\, x'^{i_1} \cdots x'^{i_k},$$

where $\bar{\partial}_i f$ on the lhs denotes the derivative of f considered as a distribution, while $\partial_i f$ on the rhs denotes the derivative of f considered as a function (which is computed using the standard rules of differentiation), Σ is any smooth close surface surrounding the origin and $\mathrm{d}\sigma_i$ is the surface element on Σ.

- The distributional derivative does not obey the Leibniz's rule. Let us suppose that it does, then

$$\bar{\partial}_i \frac{1}{r^3} = \bar{\partial}_i \left(\frac{1}{r} \frac{1}{r^2} \right) = \frac{1}{r^2}\, \bar{\partial}_i \frac{1}{r} + \frac{1}{r}\, \bar{\partial}_i \frac{1}{r^2}.$$

But the rhs can be computed using standard differential calculus (no terms with $\delta^3(\mathbf{x})$), whereas computing the lhs one obtains some term proportional to $\partial_i \delta^3(\mathbf{x})$.

Looking for the Correct 3-Dimensional Regularization (4/4)

- The Riesz-implemented Hadamard regularization is based on the Hadamard "partie finie" and the Riesz analytic continuation procedures; it relies on multiplying the full integrand, say $i(\mathbf{x})$, of the divergent integral by two regularization factors ($r_1 := |\mathbf{x} - \mathbf{x}_1|$, $r_2 := |\mathbf{x} - \mathbf{x}_2|$),

$$i(\mathbf{x}) \longrightarrow i(\mathbf{x}) \left(\frac{r_1}{s_1}\right)^{\epsilon_1} \left(\frac{r_2}{s_2}\right)^{\epsilon_2},$$

  and studying the double limit $\epsilon_1 \to 0$, $\epsilon_2 \to 0$ within analytic continuation in the complex ϵ_1 and ϵ_2 planes (here s_1 and s_2 are arbitrary 3-dimensional UV regularization scales).

- The result of employing the 3-dimensional regularization procedures described above is ambiguous—it depends on the way one writes integrands when transforming them using integration by parts (both in space and in time).

Contact Terms in d Dimensions: Hadamard's "Partie Finie"

- "Partie finie", i.e., the finite part (FP) of a singular function:

$$S(\mathbf{x}_a + r_a \mathbf{n}_a) = \sum_{m=-m_{\max}}^{\infty} a_m(\mathbf{x}_a)\, r_a^m, \quad S_{\text{reg}}(\mathbf{x}_a) \equiv \mathrm{FP}_a\, S := \frac{1}{\Omega_{d-1}} \oint \mathrm{d}\Omega_{d-1}\, a_0(\mathbf{n}_a),$$

  Ω_{d-1} is the volume of the $(d-1)$-dimensional unit sphere.

- Regularization of contact terms:

$$\Big\{ S(\mathbf{x})\, \delta^d(\mathbf{x}-\mathbf{x}_a) \Big\}_{\text{reg}} = (\mathrm{FP}_a\, S)\delta^d(\mathbf{x}-\mathbf{x}_a),$$

$$\left\{ \int \mathrm{d}^d\mathbf{x}\, S(\mathbf{x})\, \delta^d(\mathbf{x}-\mathbf{x}_a) \right\}_{\text{reg}} := \mathrm{FP}_a\, S.$$

- Important feature (in general not valid in $d = 3$ dimensions):

$$\mathrm{FP}_a\big(f_1 f_2 \cdots\big) = \big(\mathrm{FP}_a\, f_1\big)\big(\mathrm{FP}_a\, f_2\big) \cdots .$$

3PN-Related Example

- d-dimensional Newtonian potential,

$$\phi_{(2)} = -\sum_a m_a \, \Delta^{-1}\delta_a = \kappa \sum_a m_a r_a^{2-d}.$$

- 3PN Hamiltonian contains the term

$$\int \mathrm{d}^d\mathbf{x} \left(\phi_{(2)}(\mathbf{x})\right)^4 \delta^d(\mathbf{x}-\mathbf{x}_1) = \int \mathrm{d}^d\mathbf{x} \, \kappa^4 \left(m_1 r_1^{2-d} + m_2 r_2^{2-d}\right)^4 \delta^d(\mathbf{x}-\mathbf{x}_1),$$

$$\Re(d) < 2 \Rightarrow \lim_{\mathbf{x}\to\mathbf{x}_1} r_1^{2-d} = 0, \quad \text{therefore}$$

$$\int \mathrm{d}^d\mathbf{x} \left(\phi_{(2)}(\mathbf{x})\right)^4 \delta^d(\mathbf{x}-\mathbf{x}_1) = \kappa^4 \left(m_2 r_{12}^{2-d}\right)^4 = \left(\mathrm{FP}_1 \, \phi_{(2)}\right)^4.$$

- In other words in d dimensions

$$\mathrm{FP}_1 \left(\phi_{(2)}^4\right) = \left(\mathrm{FP}_1 \, \phi_{(2)}\right)^4.$$

- One easily cheks that in 3 dimensions

$$\mathrm{FP}_1 \left(\phi_{(2)}^4\right) \neq \left(\mathrm{FP}_1 \, \phi_{(2)}\right)^4.$$

Distributional Differentiation of Homogeneous Functions in d Dimensions

The 3-dimensional formula for distributional differentiation of homogeneous functions is valid (without any change) also in the d-dimensional case. E.g., it gives

$$\frac{\bar{\partial}^2}{\bar{\partial} x^i \bar{\partial} x^j}\left(\frac{1}{r_a^{d-2}}\right) = \left\{\frac{\partial^2}{\partial x^i \partial x^j}\left(\frac{1}{r_a^{d-2}}\right)\right\}_{\text{ordinary}} - \frac{4\pi^{d/2}}{d\,\Gamma(d/2-1)}\,\delta_{ij}\,\delta^d(\mathbf{x}-\mathbf{x}_a)$$

$$= (d-2)\frac{d\,n_a^i n_a^j - \delta_{ij}}{r_a^d} - \frac{4\pi^{d/2}}{d\,\Gamma(d/2-1)}\,\delta_{ij}\,\delta^d(\mathbf{x}-\mathbf{x}_a),$$

$$\Delta\left(\frac{1}{r_a^{d-2}}\right) = -\frac{4\pi^{d/2}}{\Gamma(d/2-1)}\,\delta^d(\mathbf{x}-\mathbf{x}_a),$$

$$\left(\text{in } d=3\text{:}\quad \Delta\left(\frac{1}{r_a}\right) = -4\pi\,\delta^3(\mathbf{x}-\mathbf{x}_a)\right).$$

Riesz Kernel

- Instead of d-dimensional Dirac distributions one can try to use d-dimensional Riesz kernels:

$$\delta^d(\mathbf{x} - \mathbf{x}_a) = \lim_{\varepsilon_a \to 0^+} \delta_{\varepsilon_a}(\mathbf{x} - \mathbf{x}_a),$$

$$\text{where} \quad \delta_{\varepsilon_a}(\mathbf{x} - \mathbf{x}_a) := \frac{\Gamma((d - \varepsilon_a)/2)}{\pi^{d/2}\, 2^{\varepsilon_a}\, \Gamma(\varepsilon_a/2)}\, r_a^{\varepsilon_a - d}.$$

- One replaces in the constraint equations Dirac-δ-sources by Riesz kernels, solves the constraints pertubatively and develop the whole PN scheme.
- At the end of computations, one takes the limits $\varepsilon_1 \to 0^+$, $\varepsilon_2 \to 0^+$, and only after this one computes $d \to 3$ limit.
- No distributional differentiations are needed.
- The usage of the Riesz kernel directly in 3 dimensions does not resolve ambiguities.

The Extended Hadamard Regularization (1/2)

- The extended Hadamard regularization (EHR) is a specific variant of 3-dimensional Hadamard regularization devised by Blanchet & Faye and used by them in computation of the 3PN two-point-mass EOM in harmonic coordinates (2000–01).
- The basic idea is to associate to any function $F \in \mathcal{F}$, where the set $\mathcal{F}$ comprises functions smooth on $\mathbb{R}^3$ except for the two points (around which they admit a power-like singular expansion) a pseudo-function $\mathrm{Pf}F$, which is a linear form acting on functions from $\mathcal{F}$:

$$\langle \mathrm{Pf}F, G \rangle := \mathrm{Pf}_{s_1,s_2} \int \mathrm{d}^3x \, FG, \quad \text{for any } G \in \mathcal{F},$$

  where Pf_{s_1,s_2} means partie finie of the divergent integral (it depends on two—one per each singularity—arbitrary regularization scales s_1 and s_2).
- The Dirac δ-functions δ_a are represented by the pseudo-functions $\mathrm{Pf}\delta_a$ defined by

$$\langle \mathrm{Pf}\delta_a, G \rangle := G_{\mathrm{reg}}(\mathbf{x}_a), \quad \text{for any } G \in \mathcal{F},$$

  The product $F\delta_a$ is represented by another pseudo-function $\mathrm{Pf}(F\delta_a)$:

$$\langle \mathrm{Pf}(F\delta_a), G \rangle := (FG)_{\mathrm{reg}}(\mathbf{x}_a), \quad \text{for any } G \in \mathcal{F}.$$

  As a consequence, in general $\mathrm{Pf}(F\delta_a) \neq F_{\mathrm{reg}}(\mathbf{x}_a)\mathrm{Pf}\delta_a$.
- To ensure the possibility of integration by parts, partial derivatives of singular functions are specifically treated. This leads to a distributional derivative, which differs in general from the Schwartz derivative. E.g.,

$$\bar{\partial}_i \mathrm{Pf}\frac{1}{r} = -\mathrm{Pf}\frac{n^i}{r^2} + 2\pi \mathrm{Pf}(rn^i\delta), \quad \text{Schwartz derivative gives} \quad \bar{\partial}_i \frac{1}{r} = -\frac{n^i}{r^2}.$$

The Extended Hadamard Regularization (2/2)

The definitions adopted by EHR disagree with DR rules.

- In generic d dimensions one can always use

$$F^{(d)}(\mathbf{x})\delta^d\left(\mathbf{x}-\mathbf{x}_a\right) = F^{(d)}_{\mathrm{reg}}(\mathbf{x}_a)\,\delta^d(\mathbf{x}-\mathbf{x}_a),$$

  where $F^{(d)}$ is the d-dimensional version of 3-dimensional F. This leads to the following DR rule, which disagrees with the EHR rule:

$$\left[F(\mathbf{x})\,\delta^3(\mathbf{x}-\mathbf{x}_a)\right]_{\mathrm{reg}} := \left(\lim_{d\to 3} F^{(d)}_{\mathrm{reg}}(\mathbf{x}_a)\right)\delta^3(\mathbf{x}-\mathbf{x}_a).$$

- The EHR differentiation when applied to smooth functions of compact support, coincides with Schwartz differentiation. However, in the 3PN-level computations it operated with other singular functions and gave the results different from the results obtained by applying Schwartz differentiation.
  The definition of Schwartz differentiation is valid in d dimensions, what supports the use of this definition also in three dimensions.

- The computation using EHR can not be combined with DR. This can be seen from DR completion of the 3PN EOM in harmonic coordinates: before applying DR it was necessary to subtract all contributions, which were direct consequences of the use of EHR.
  However, at the 3PN level the difference between the final results of EHR and DR computations of two-point-mass EOM can be described in terms of one dimensionless ambiguity parameter.

Looking for the Proper Modification of the Schwartz Theory?

- Inspired by the EHR of Blanchet & Faye, mathematicians have recently developed the theory of "thick distributions": Estrada & Fulling (2007) in one dimension, and Yang & Estrada (2013) in higher dimensions. This theory is connected with the EHR, but is not equivalent to the latter and it can not be used to improve regularization issues in the PN two-body problem.

> *(...) it is not correct to say that the work of Laurent Schwartz justifies everything that physicists do with the Dirac delta function, because sometimes they do things that are clearly wrong. There is a spectrum of responses to this situation. The first (chosen by too many mathematicians) is to dismiss distributions as untrustworthy, a kind of pornography that should be kept out of the hands of engineering and science students. Another (adopted by many practitioners) is to rationalize after the fact whatever interpretation of the symbols gives the right answer in the problem at hand (...) sometimes this is done in blatant contradiction to interpretations adopted in other contexts. A safer approach is to regard the delta function as a heuristic device that leads rapidly to formulas whose correctness must then be rigorously verified (e.g., by substituting a putative solution back into a differential equation). But one cannot be satisfied just with this; if distributions are unambiguously defined as linear functionals on spaces of test functions, then their properties must be unambiguous, and the mathematician should determine which formulas and calculational rules are true and why—tightening up the definitions when necessary.*
>
> [R.Estrada & S.A.Fulling, Int. J. Appl. Math. Stat. **10**, 25 (2007)]

Field-Like Terms

- In d dimensions:

$$\int r_1^{\alpha} r_2^{\beta}\, \mathrm{d}^d\mathbf{x} = \pi^{d/2} \frac{\Gamma\left(\frac{\alpha+d}{2}\right)\Gamma\left(\frac{\beta+d}{2}\right)\Gamma\left(-\frac{\alpha+\beta+d}{2}\right)}{\Gamma\left(-\frac{\alpha}{2}\right)\Gamma\left(-\frac{\beta}{2}\right)\Gamma\left(\frac{\alpha+\beta+2d}{2}\right)}\, r_{12}^{\alpha+\beta+d}.$$

- In $d = 3$ dimensions (Jaranowski & Schäfer 1998):

$$\begin{aligned}\int r_1^{\alpha} r_2^{\beta} (r_1 + r_2 + r_{12})^{\gamma}\, \mathrm{d}^3\mathbf{x} = 2\pi\, r_{12}^{\alpha+\beta+\gamma+3} \\ \times \Big(B(\alpha+2, \beta+2)\, B_{1/2}(-\alpha-\beta-\gamma-4, \alpha+\beta+4) \\ - B(-\alpha-\beta-4, \beta+2)\, B_{1/2}(-\alpha-\gamma-2, \alpha+2) \\ - B(-\alpha-\beta-4, \alpha+2)\, B_{1/2}(-\beta-\gamma-2, \beta+2) \Big),\end{aligned}$$

where B is the beta function and $B_{1/2}$ is the incomplete beta function:

$$B_{1/2}(\alpha, \beta) = \frac{1}{\alpha\, 2^{\alpha}}\, {}_2F_1\left(1-\beta, \alpha; \alpha+1; \frac{1}{2}\right),$$

${}_2F_1$ is the Gauss hypergeometric function.

Field-like terms: Riesz-Implemented Hadamard Regularization

- Let the integrand $i(\mathbf{x})$ develops only local poles, then its RH-regularized value reads

$$I^{\mathrm{RH}}(3;\epsilon_1,\epsilon_2) := \int_{\mathbb{R}^3} i(\mathbf{x}) \left(\frac{r_1}{s_1}\right)^{\epsilon_1} \left(\frac{r_2}{s_2}\right)^{\epsilon_2} \mathrm{d}^3\mathbf{x}$$

$$= A + c_1\left(\frac{1}{\epsilon_1} + \ln\frac{r_{12}}{s_1}\right) + c_2\left(\frac{1}{\epsilon_2} + \ln\frac{r_{12}}{s_2}\right) + \mathcal{O}(\epsilon_1,\epsilon_2),$$

  s_1 and s_2 are arbitrary UV regularization scales.

- The pole, say $\propto 1/\epsilon_1$, comes from the part of the integrand $i(\mathbf{x})$ which develops logarithmic singularities (i.e. locally behaves like $1/r_1^3$),

$$i(\mathbf{x}) = \cdots + \tilde{c}_1(\mathbf{n}_1)\, r_1^{-3} + \cdots\,, \quad \text{when } \mathbf{x} \to \mathbf{x}_1.$$

  The pole part can be recovered by RH regularization of the integral of $\tilde{c}_1(\mathbf{n}_1)\, r_1^{-3}$ over the ball $B(\mathbf{x}_1,\ell_1)$:

$$I_1^{\mathrm{RH}}(3;\epsilon_1) := \int_{B(\mathbf{x}_1,\ell_1)} \tilde{c}_1(\mathbf{n}_1)\, r_1^{-3} \left(\frac{r_1}{s_1}\right)^{\epsilon_1} \mathrm{d}^3\mathbf{r}_1 = c_1\left(\frac{1}{\epsilon_1} + \ln\frac{\ell_1}{s_1}\right) + \mathcal{O}(\epsilon_1).$$

Field-Like Terms: Implementation of DR

- It is enough to replace $I_1^{\mathrm{RH}}(3;\epsilon_1)$ and $I_2^{\mathrm{RH}}(3;\epsilon_2)$ by their d-dimensional versions $I_1(d,\ell_0)$ and $I_2(d,\ell_0)$, where ℓ_0 is the DR length scale.
- One considers d-dimensional version of the expansion of $i(\mathbf{x})$,

$$i(\mathbf{x}) = \cdots + \tilde{c}_1(d,\ell_0;\mathbf{n}_1)\, r_1^{6-3d} + \cdots\,, \quad \text{when } \mathbf{x}\to\mathbf{x}_1,$$

  and defines

$$I_1(d,\ell_0) := \int_{B(\mathbf{x}_1,\ell_1)} \tilde{c}_1(d,\ell_0;\mathbf{n}_1)\, r_1^{6-3d}\, \mathrm{d}^d\mathbf{r}_1 = c_1\Big(-\frac{1}{2\varepsilon} + \ln\frac{\ell_1}{\ell_0}\Big) + B_1 + \mathcal{O}(\varepsilon).$$

- The DR correction to the RH-regularized integral $I^{\mathrm{RH}}(3;\epsilon_1,\epsilon_2)$:

$$\begin{aligned} I^{\mathrm{RH}}(3;\epsilon_1,\epsilon_2) - I_1^{\mathrm{RH}}(3;\epsilon_1) - I_2^{\mathrm{RH}}(3;\epsilon_2) + I_1(d,\ell_0) + I_2(d,\ell_0) \\ = A + \Delta A - \frac{c_1+c_2}{2\varepsilon} + B\ln\frac{r_{12}}{\ell_0} + \mathcal{O}(\varepsilon). \end{aligned}$$

  The result is as if all computations were fully done in d dimensions.

- One looks for the local behavior, say, around $\mathbf{x} = \mathbf{x}_1$, of the solution of equation

$$\Delta f(\mathbf{x}) = g(\mathbf{x} - \mathbf{x}_1, \mathbf{x} - \mathbf{x}_2).$$

- One expands the source term g around $r_1 = |\mathbf{x} - \mathbf{x}_1|$:

$$g(\mathbf{x} - \mathbf{x}_1, \mathbf{x} - \mathbf{x}_2) = g(r_1\mathbf{n}_1, r_1\mathbf{n}_1 + r_{12}\mathbf{n}_{12}) = \sum_{k=-m}^{\infty} g_k(\mathbf{n}_1) r_1^k,$$

  where $m \geq 0$ is some nonnegative integer. After applying the operator Δ^{-1} to each term of the expansion one gets the expansion of the solution near $\mathbf{x} = \mathbf{x}_1$:

$$f_{\text{nonhom}}(\mathbf{x}) = \sum_{k=-m}^{\infty} \Delta^{-1}\big(g_k(\mathbf{n}_1) r_1^k\big).$$

- The formal solution of the Poisson equation has the form of the integral

$$f(\mathbf{x}) = -\kappa \int \mathrm{d}^d\mathbf{x}' g(\mathbf{x}') |\mathbf{x} - \mathbf{x}'|^{2-d}.$$

  One expands the kernel around $r_1 = 0$ and integrates the sum term by term,

$$|\mathbf{x} - \mathbf{x}'|^{2-d} = |r_1\mathbf{n}_1 - r_1'\mathbf{n}_1'|^{2-d} = \sum_{\ell=0}^{\infty} K_\ell(\mathbf{n}_1; \mathbf{n}_1', r_1') r_1^\ell.$$

$$f_{\text{hom}}(\mathbf{x}) = -\kappa \sum_{\ell=0}^{\infty} \int \mathrm{d}^d\mathbf{x}' g(\mathbf{x}') K_\ell(\mathbf{n}_1; \mathbf{n}_1', r_1') r_1^\ell.$$

Example (1/2)

- The distributional differentiation is necessary when one differentiates homogeneous functions under the integral sign. Let us consider the following locally divergent integral:

$$p_{1i}\, p_{1j} \int \mathrm{d}^3\mathbf{x} \left(\partial_i \partial_j \frac{1}{r_1}\right) \frac{1}{r_2^4}.$$

  We shall regularize this integral in two different ways.

- We first replace in the integrand differentiations with respect to x^i by those with respect to x_1^i (obviously $\partial_i r_1 = -\partial_{1i} r_1$). Then we shift the differentiations before the integral sign and apply directly the Riesz formula. The result is

$$p_{1i}\, p_{1j} \int \mathrm{d}^3\mathbf{x} \left(\partial_i \partial_j \frac{1}{r_1}\right) \frac{1}{r_2^4} = p_{1i}\, p_{1j}\, \partial_{1i}\partial_{1j} \int \frac{\mathrm{d}^3 x}{r_1 r_2^4}$$

$$= p_{1i}\, p_{1j}\, \partial_{1i}\partial_{1j} \left(-\frac{2\pi}{r_{12}^2}\right) = \frac{4\pi\left[\mathbf{p}_1^2 - 4(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\right]}{r_{12}^4}.$$

  We have obtained this result performing integration first and then differentiation.

Example (2/2)

- Now we shall regularize the integral doing differentiation first. To do it we have to use the distributional differentiation, which gives

$$\partial_i\partial_j \frac{1}{r_1} = \left(3n_1^i n_1^j - \delta_{ij}\right)\frac{1}{r_1^3} - \frac{4\pi}{3}\delta_{ij}\delta^3(\mathbf{x}-\mathbf{x}_1).$$

We substitute this into the integral:

$$p_{1i}\,p_{1j}\int \mathrm{d}^3\mathbf{x}\left(\partial_i\partial_j\frac{1}{r_1}\right)\frac{1}{r_2^4} = \int \mathrm{d}^3x\,\frac{3(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2-\mathbf{p}_1^2}{r_1^3 r_2^4} - \frac{4\pi}{3}\mathbf{p}_1^2\int \mathrm{d}^3\mathbf{x}\,\frac{1}{r_2^4}\delta^3(\mathbf{x}-\mathbf{x}_1).$$

The second integral on the RHS is obviously regularized to

$$\int \mathrm{d}^3\mathbf{x}\,\frac{1}{r_2^4}\delta^3(\mathbf{x}-\mathbf{x}_1) = \frac{1}{r_{12}^4}.$$

To calculate the first integral on the RHS we apply 3-dimensional Riesz-implemented Hadamard regularization. We obtain

$$\int \mathrm{d}^3\mathbf{x}\,\frac{3(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2-\mathbf{p}_1^2}{r_1^3 r_2^4} = \frac{16\pi\left[\mathbf{p}_1^2-3(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\right]}{3r_{12}^4}.$$

Collecting all the results together we get the result, which coincides with the result obtained before.

- The two ways of regularizing the integral, described above, give the same result only if we apply distributional differentiation when we perform differentiation before integration.

UV Regularization of 3PN/4PN Hamiltonians

- Regularization of the 3PN Hamiltonian:

$$\Delta H^{\text{DR correction}}_{\text{3PN}}(\mathbf{x}_a, \mathbf{p}_a; \varepsilon) = C(\mathbf{x}_a, \mathbf{p}_a) + \mathcal{O}(\varepsilon) \quad (\text{no pole part!}),$$

$$\text{Reg}_{\text{UV}}\{H_{\text{3PN}}(\mathbf{x}_a, \mathbf{p}_a)\} := \lim_{\varepsilon\to 0}\left\{H^{\text{RH}}_{\text{3PN}}(\mathbf{x}_a, \mathbf{p}_a) + \Delta H^{\text{DR correction}}_{\text{3PN}}(\mathbf{x}_a, \mathbf{p}_a; \varepsilon)\right\}.$$

- Regularization of the near-zone IR-convergent part of the 4PN Hamiltonian:

$$\Delta H^{\text{DR correction}}_{\text{4PN}}(\mathbf{x}_a, \mathbf{p}_a; \varepsilon) = C_1(\mathbf{x}_a, \mathbf{p}_a) + \frac{C_2(\mathbf{x}_a, \mathbf{p}_a)}{\varepsilon} + C_3(\mathbf{x}_a, \mathbf{p}_a)\ln\frac{r_{12}}{\ell_0} + \mathcal{O}(\varepsilon).$$

One finds *unique* $D_1(\mathbf{x}_a, \mathbf{p}_a)$ and $D_1(\mathbf{x}_a, \mathbf{p}_a)$ such that the following limit is finite:

$$\begin{aligned}\text{Reg}_{\text{UV}}\{H^{\text{near-zone}}_{\text{4PN, IR conv}}(\mathbf{x}_a, \mathbf{p}_a)\} := \lim_{\varepsilon\to 0}\Big\{&H^{\text{RH}}_{\text{4PN, IR conv}}(\mathbf{x}_a, \mathbf{p}_a)\\ &+\Delta H^{\text{DR correction}}_{\text{4PN}}(\mathbf{x}_a, \mathbf{p}_a; \varepsilon)\\ &+\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{D_1(\mathbf{x}_a, \mathbf{p}_a)}{\varepsilon} + D_2(\mathbf{x}_a, \mathbf{p}_a)\ln\frac{r_{12}}{\ell_0}\Big)\Big\}.\end{aligned}$$

Regularization of the Near-Zone IR Divergences (1/2)

- All terms generating IR divergences have the following structure

$$f_{ij}(\mathbf{x})\,\Delta^{-1}\left(\ddot{h}^{\mathrm{TT}}_{(4)ij}\right) = f_{ij}(\mathbf{x})\,\Delta^{-2}\,\partial_t^2\,S^{\mathrm{TT}}_{(4)ij}.$$

  They develop the logarithmic divergences linked to a decay of the integrand $\propto r^{-3-3(d-3)}$ as $r := |\mathbf{x}| \to \infty$.

- Damour/Jaranowski/Schäfer (2014–15) used two methods to perform IR regularization. In both methods one introduces a new IR regularization length scale s.
  - — Modifying behavior of the $h^{\mathrm{TT}}_{(6)ij}$ at spatial infinity: in all the IR-divergent terms one can make the replacement

$$\Delta^{-1}\left[\ddot{h}^{\mathrm{TT}}_{(4)ij}\right] \to \Delta^{-1}\left[\left(\frac{r}{s}\right)^{B}\,\ddot{h}^{\mathrm{TT}}_{(4)ij}\right]^{\mathrm{TT}},$$

    and then take the finite part of the IR pole at $B = 2(d-3)$.
  - — d-dimensional version of the Riesz-implemented Hadamard regularization: one multiplies, before integrating it over space, the full integrand by a factor

$$\left(\frac{r_1}{s}\right)^{\alpha}\left(\frac{r_2}{s}\right)^{\beta},$$

    and take the finite part of the IR pole occurring at $\alpha + \beta = 2(d-3)$.

Regularization of the Near-Zone IR Divergences (2/2)

- Both methods yield the same result modulo a time derivative and a change in some constant C,

$$\mathrm{Reg}^{s}_{\mathrm{IR}}\{H^{\text{near-zone}}_{\text{4PN, IR div}}\}(\mathbf{x}_a, \mathbf{p}_a; C) = \bar{\chi}(\mathbf{x}_a, \mathbf{p}_a) + \frac{2}{5}\,\frac{G^2 M}{c^8}\left(\dddot{I}_{ij}\right)^2 \left(\ln\frac{r_{12}}{s} + C\right) + \frac{\mathrm{d}}{\mathrm{d}t}G(\mathbf{x}_a, \mathbf{p}_a),$$

  The constant C enters the Hamiltonian through the term $\ln(r_{12}/s) + C$.

- The addition of the constant C to the logarithm $\ln(r_{12}/s)$ is related to the arbitrariness of the IR-regularization scale s:
  replacing s by $s' = e^{-\lambda}s$ is equivalent to replacing C by $C' = C + \lambda$.

The Total 2-Point-Mass Conservative 4PN Hamiltonian

- After adding the IR-convergent part (and dropping the total time derivative) and the tail contribution one gets the total 4PN Hamiltonian,

$$H_{4\mathrm{PN}}[\mathbf{x}_a, \mathbf{p}_a; C] = \mathrm{Reg}_{\mathrm{UV}}\{H^{\text{near-zone}}_{\text{4PN, IR conv}}\}(\mathbf{x}_a, \mathbf{p}_a) + \mathrm{Reg}^{s}_{\mathrm{IR}}\{H^{\text{near-zone}}_{\text{4PN, IR div}}\}(\mathbf{x}_a, \mathbf{p}_a; C)$$
$$+ \mathrm{Reg}^{s}\{H^{\text{tail sym}}_{4\mathrm{PN}}\}[\mathbf{x}_a, \mathbf{p}_a]$$
$$= \chi(\mathbf{x}_a, \mathbf{p}_a) + \frac{2}{5}\frac{G^2 M}{c^8}(\dddot{I}_{ij})^2\left(\ln\frac{r_{12}}{s} + C\right)$$
$$- \frac{1}{5}\frac{G^2 M}{c^8}\dddot{I}_{ij}\,\mathrm{Pf}_{2s/c}\int_{-\infty}^{+\infty}\frac{\mathrm{d}v}{|v|}\dddot{I}_{ij}(t+v).$$

- Because

$$-\frac{1}{5}\frac{G^2 M}{c^8}\dddot{I}_{ij}\,\mathrm{Pf}_{2s/c}\int_{-\infty}^{+\infty}\frac{\mathrm{d}v}{|v|}\dddot{I}_{ij}(t+v) = +\frac{2}{5}\frac{G^2 M}{c^8}(\dddot{I}_{ij})^2\ln(2s/c) + \cdots,$$

the dependence on s cancels between the near-zone and tail contributions, and the total 4PN Hamiltonian reads

$$H_{4\mathrm{PN}}[\mathbf{x}_a, \mathbf{p}_a; C] = \chi(\mathbf{x}_a, \mathbf{p}_a) + \frac{2}{5}\frac{G^2 M}{c^8}(\dddot{I}_{ij})^2\, C$$
$$- \frac{1}{5}\frac{G^2 M}{c^8}\dddot{I}_{ij}\,\mathrm{Pf}_{2r_{12}/c}\int_{-\infty}^{+\infty}\frac{\mathrm{d}v}{|v|}\dddot{I}_{ij}(t+v).$$

Determination of the Value of the Constant C: Using Beyond-Near-Zone Information

- One needs a calculation which takes into account the transition between the near zone and the wave zone without losing any information.
  Such a calculation was performed by Bini & Damour (2013)
  within the gravitational self-force approach
  in the case of the dynamics of circular orbits
  and in the first order in the symmetric-mass-ratio $\nu := m_1 m_2/(m_1+m_2)^2$.
- It is enough to consider the 4PN-accurate gauge-invariant link between the binding energy $E := H - Mc^2$ and the angular momentum $j := cJ/(Gm_1m_2)$ along circular orbits,

$$E_{\leq 4\mathrm{PN}}(j;\nu) = -\frac{1}{2}\mu c^2 \frac{1}{j^2}\left(1 + \frac{a_{1\mathrm{PN}}(\nu)}{j^2} + \cdots + \frac{a^1_{4\mathrm{PN}}(\nu) + a^2_{4\mathrm{PN}}(\nu)\ln j}{j^8}\right).$$

- The comparison of the linear in ν 4PN-level term predicted by $H_{4\mathrm{PN}}[\mathbf{x}_a, \mathbf{p}_a; C]$ to that derived from the results of Bini & Damour (2013)
  yields the unique value of C ($C = -1681/1536$).
  This completed the determination of the 4PN conservative dynamics of 2-point-mass systems.

4PN-Accurate Conservative 2-Point-Mass ADM Hamiltonian

$$H_{\leq 4\mathrm{PN}}[\mathrm{x}_a, \mathrm{p}_a] = H^{\mathrm{local}}_{\leq 4\mathrm{PN}}(\mathrm{x}_a, \mathrm{p}_a) + H^{\mathrm{nonlocal}}_{4\mathrm{PN}}[\mathrm{x}_a, \mathrm{p}_a] \quad (a = 1, 2).$$

Local-in-time 4PN-accurate Hamiltonian

$$H^{\mathrm{local}}_{\leq 4\mathrm{PN}}(\mathrm{x}_a, \mathrm{p}_a) = (m_1 + m_2)c^2 + H_{\mathrm{N}}(\mathrm{x}_a, \mathrm{p}_a) + \frac{1}{c^2} H_{1\mathrm{PN}}(\mathrm{x}_a, \mathrm{p}_a) + \frac{1}{c^4} H_{2\mathrm{PN}}(\mathrm{x}_a, \mathrm{p}_a) + \frac{1}{c^6} H_{3\mathrm{PN}}(\mathrm{x}_a, \mathrm{p}_a) + \frac{1}{c^8} H^{\mathrm{local}}_{4\mathrm{PN}}(\mathrm{x}_a, \mathrm{p}_a).$$

Nonlocal-in-time 4PN Hamiltonian

(Blanchet-Damour 1988, Damour-Jaranowski-Schäfer 2014)

$$H^{\mathrm{nonlocal}}_{4\mathrm{PN}}[\mathrm{x}_a, \mathrm{p}_a] = -\frac{1}{5}\frac{G^2 M}{c^8} \dddot{I}_{ij} \times \mathrm{Pf}_{2r_{12}/c} \left(\int_{-\infty}^{+\infty} \frac{\mathrm{d}v}{|v|} \dddot{I}_{ij}(t+v) \right).$$

Newtonian/1PN/2PN Hamiltonians

The operation "$+(1 \leftrightarrow 2)$" used below denotes the addition for each term of another term obtained by the label permutation $1 \leftrightarrow 2$.

$$H_{\rm N}(\mathbf{x}_a, \mathbf{p}_a) = \frac{\mathbf{p}_1^2}{2m_1} - \frac{Gm_1 m_2}{2r_{12}} + (1 \leftrightarrow 2),$$

$$H_{1\rm PN}(\mathbf{x}_a, \mathbf{p}_a) = -\frac{(\mathbf{p}_1^2)^2}{8m_1^3} + \frac{Gm_1 m_2}{4r_{12}}\left(-6\frac{\mathbf{p}_1^2}{m_1^2} + 7\frac{(\mathbf{p}_1 \cdot \mathbf{p}_2)}{m_1 m_2} + \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_1 m_2}\right) + \frac{G^2 m_1^2 m_2}{2r_{12}^2} + (1 \leftrightarrow 2),$$

$$\begin{aligned} H_{2\rm PN}(\mathbf{x}_a, \mathbf{p}_a) = {} & \frac{1}{16}\frac{(\mathbf{p}_1^2)^3}{m_1^5} + \frac{1}{8}\frac{Gm_1 m_2}{r_{12}}\left(5\frac{(\mathbf{p}_1^2)^2}{m_1^4} - \frac{11}{2}\frac{\mathbf{p}_1^2\,\mathbf{p}_2^2}{m_1^2 m_2^2} - \frac{(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{m_1^2 m_2^2} + 5\frac{\mathbf{p}_1^2\,(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{m_1^2 m_2^2} - 6\frac{(\mathbf{p}_1 \cdot \mathbf{p}_2)\,(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_1^2 m_2^2} - \frac{3}{2}\frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{m_1^2 m_2^2}\right) \\ & + \frac{1}{4}\frac{G^2 m_1 m_2}{r_{12}^2}\left(m_2\left(10\frac{\mathbf{p}_1^2}{m_1^2} + 19\frac{\mathbf{p}_2^2}{m_2^2}\right) - \frac{1}{2}(m_1 + m_2)\frac{27\,(\mathbf{p}_1 \cdot \mathbf{p}_2) + 6\,(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_1 m_2}\right) - \frac{1}{8}\frac{Gm_1 m_2}{r_{12}}\frac{G^2(m_1^2 + 5m_1 m_2 + m_2^2)}{r_{12}^2} + (1 \leftrightarrow 2). \end{aligned}$$

3PN Hamiltonian (Damour-Jaranowski-Schäfer 2001)

$$\begin{aligned} H_{3\rm PN}(\mathbf{x}_a, \mathbf{p}_a) = {} & -\frac{5}{128}\frac{(\mathbf{p}_1^2)^4}{m_1^7} + \frac{1}{32}\frac{Gm_1 m_2}{r_{12}}\left(-14\frac{(\mathbf{p}_1^2)^3}{m_1^6} + 4\frac{((\mathbf{p}_1 \cdot \mathbf{p}_2)^2 + 4\,\mathbf{p}_1^2\,\mathbf{p}_2^2)\mathbf{p}_1^2}{m_1^4 m_2^2} + 6\frac{\mathbf{p}_1^2\,(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{m_1^4 m_2^2} - 10\frac{(\mathbf{p}_1^2\,(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2 + \mathbf{p}_2^2\,(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2)\mathbf{p}_1^2}{m_1^4 m_2^2}\right. \\ & + 24\frac{\mathbf{p}_1^2\,(\mathbf{p}_1 \cdot \mathbf{p}_2)(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_1^4 m_2^2} + 2\frac{\mathbf{p}_1^2\,(\mathbf{p}_1 \cdot \mathbf{p}_2)(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{m_1^3 m_2^3} + \frac{(7\,\mathbf{p}_1^2\,\mathbf{p}_2^2 - 10\,(\mathbf{p}_1 \cdot \mathbf{p}_2)^2)(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_1^3 m_2^3} + \frac{(\mathbf{p}_1^2\,\mathbf{p}_2^2 - 2\,(\mathbf{p}_1 \cdot \mathbf{p}_2)^2)(\mathbf{p}_1 \cdot \mathbf{p}_2)}{m_1^3 m_2^3} \\ & \left. + 15\frac{(\mathbf{p}_1 \cdot \mathbf{p}_2)(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{m_1^3 m_2^3} - 18\frac{\mathbf{p}_1^2\,(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)^3}{m_1^3 m_2^3} + 5\frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^3(\mathbf{n}_{12} \cdot \mathbf{p}_2)^3}{m_1^3 m_2^3}\right) + \frac{G^2 m_1 m_2}{r_{12}^2}\left(\frac{1}{16}(m_1 - 27m_2)\frac{(\mathbf{p}_1^2)^2}{m_1^4}\right. \\ & - \frac{115}{16}m_1\frac{\mathbf{p}_1^2\,(\mathbf{p}_1 \cdot \mathbf{p}_2)}{m_1^3 m_2} + \frac{1}{48}m_2\frac{25\,(\mathbf{p}_1 \cdot \mathbf{p}_2)^2 + 371\,\mathbf{p}_1^2\,\mathbf{p}_2^2}{m_1^2 m_2^2} + \frac{17}{16}\frac{\mathbf{p}_1^2(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2}{m_1^3} + \frac{5}{12}\frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^4}{m_1^3} - \frac{3}{2}m_1\frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^3(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_1^3 m_2} \\ & - \frac{1}{8}m_1\frac{(15\,\mathbf{p}_1^2\,(\mathbf{n}_{12} \cdot \mathbf{p}_2) + 11\,(\mathbf{p}_1 \cdot \mathbf{p}_2)\,(\mathbf{n}_{12} \cdot \mathbf{p}_1))(\mathbf{n}_{12} \cdot \mathbf{p}_1)}{m_1^3 m_2} + \frac{125}{12}m_2\frac{(\mathbf{p}_1 \cdot \mathbf{p}_2)\,(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_1^2 m_2^2} + \frac{10}{3}m_2\frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{m_1^2 m_2^2} \\ & \left. - \frac{1}{48}(220m_1 + 193m_2)\frac{\mathbf{p}_1^2(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{m_1^2 m_2^2}\right) + \frac{G^3 m_1 m_2}{r_{12}^3}\left(-\frac{1}{48}\left(425\,m_1^2 + \left(473 - \frac{3}{4}\pi^2\right)m_1 m_2 + 150\,m_2^2\right)\frac{\mathbf{p}_1^2}{m_1^2}\right. \\ & + \frac{1}{16}\left(77(m_1^2 + m_2^2) + \left(143 - \frac{1}{4}\pi^2\right)m_1 m_2\right)\frac{(\mathbf{p}_1 \cdot \mathbf{p}_2)}{m_1 m_2} + \frac{1}{16}\left(20\,m_1^2 - \left(43 + \frac{3}{4}\pi^2\right)m_1 m_2\right)\frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2}{m_1^2} \\ & \left. + \frac{1}{16}\left(21(m_1^2 + m_2^2) + \left(119 + \frac{3}{4}\pi^2\right)m_1 m_2\right)\frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_1 m_2}\right) + \frac{1}{8}\frac{G^4 m_1 m_2^3}{r_{12}^4}\left(\left(\frac{227}{3} - \frac{21}{4}\pi^2\right)m_1 + m_2\right) + (1 \leftrightarrow 2). \end{aligned}$$

4PN Local-in-time Hamiltonian (Damour-Jaranowski-Schäfer 2014)

$$
\begin{aligned}
H^{\text{local}}_{4\text{PN}}(\mathbf{x}_a,\mathbf{p}_a) ={}& \frac{7(\mathbf{p}_1^2)^5}{256m_1^9} + \frac{Gm_1m_2}{r_{12}}\Bigg(\frac{45(\mathbf{p}_1^2)^4}{128m_1^8} - \frac{9(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1^2)^2}{64m_1^6m_2^2} + \frac{15(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1^2)^3}{64m_1^6m_2^2} - \frac{9(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1^2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{16m_1^6m_2^2} - \frac{3(\mathbf{p}_1^2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{32m_1^6m_2^2} + \frac{15(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1^2)^2\mathbf{p}_2^2}{64m_1^6m_2^2} - \frac{21(\mathbf{p}_1^2)^3\mathbf{p}_2^2}{64m_1^6m_2^2} - \frac{35(\mathbf{n}_{12}\cdot\mathbf{p}_1)^5(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3}{256m_1^5m_2^3} \\
&+ \frac{25(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3\mathbf{p}_1^2}{128m_1^5m_2^3} + \frac{33(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3(\mathbf{p}_1^2)^2}{256m_1^5m_2^3} - \frac{85(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{256m_1^5m_2^3} - \frac{45(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{128m_1^5m_2^3} - \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1^2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{256m_1^5m_2^3} + \frac{25(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{64m_1^5m_2^3} \\
&+ \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{64m_1^5m_2^3} - \frac{3(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)^3}{64m_1^5m_2^3} + \frac{3\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)^3}{64m_1^5m_2^3} + \frac{55(\mathbf{n}_{12}\cdot\mathbf{p}_1)^5(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_2^2}{256m_1^5m_2^3} - \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2\mathbf{p}_2^2}{128m_1^5m_2^3} - \frac{25(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1^2)^2\mathbf{p}_2^2}{256m_1^5m_2^3} - \frac{23(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{256m_1^5m_2^3} \\
&+ \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{128m_1^5m_2^3} - \frac{7(\mathbf{p}_1^2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{256m_1^5m_2^3} - \frac{5(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^4\mathbf{p}_1^2}{64m_1^4m_2^4} + \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_2)^4(\mathbf{p}_1^2)^2}{64m_1^4m_2^4} - \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{4m_1^4m_2^4} + \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{16m_1^4m_2^4} - \frac{5(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_2^2}{64m_1^4m_2^4} \\
&+ \frac{21(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2\mathbf{p}_2^2}{64m_1^4m_2^4} - \frac{3(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1^2)^2\mathbf{p}_2^2}{32m_1^4m_2^4} - \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{4m_1^4m_2^4} + \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{16m_1^4m_2^4} + \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2\mathbf{p}_2^2}{16m_1^4m_2^4} - \frac{\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2\mathbf{p}_2^2}{32m_1^4m_2^4} + \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{p}_2^2)^2}{64m_1^4m_2^4} \\
&- \frac{3(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2(\mathbf{p}_2^2)^2}{32m_1^4m_2^4} - \frac{7(\mathbf{p}_1^2)^2(\mathbf{p}_2^2)^2}{128m_1^4m_2^4}\Bigg) + \frac{G^2m_1m_2}{r_{12}^2}\,m_1\Bigg(\frac{369(\mathbf{n}_{12}\cdot\mathbf{p}_1)^6}{160m_1^6} - \frac{889(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4\mathbf{p}_1^2}{192m_1^6} + \frac{49(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1^2)^2}{16m_1^6} - \frac{63(\mathbf{p}_1^2)^3}{64m_1^6} - \frac{549(\mathbf{n}_{12}\cdot\mathbf{p}_1)^5(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{128m_1^5m_2} + \frac{67(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2}{16m_1^5m_2} \\
&- \frac{167(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1^2)^2}{128m_1^5m_2} + \frac{1547(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{p}_1\cdot\mathbf{p}_2)}{256m_1^5m_2} - \frac{851(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{128m_1^5m_2} + \frac{1099(\mathbf{p}_1^2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{256m_1^5m_2} + \frac{3263(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{1280m_1^4m_2^2} + \frac{1067(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2}{480m_1^4m_2^2} - \frac{4567(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1^2)^2}{3840m_1^4m_2^2} \\
&- \frac{3571(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)}{320m_1^4m_2^2} + \frac{3073(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{480m_1^4m_2^2} + \frac{4349(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{1280m_1^4m_2^2} - \frac{3461\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{3840m_1^4m_2^2} + \frac{1673(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4\mathbf{p}_2^2}{1920m_1^4m_2^2} - \frac{1999(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2\mathbf{p}_2^2}{3840m_1^4m_2^2} + \frac{2081(\mathbf{p}_1^2)^2\mathbf{p}_2^2}{3840m_1^4m_2^2} \\
&- \frac{13(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3}{8m_1^3m_2^3} + \frac{191(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3\mathbf{p}_1^2}{192m_1^3m_2^3} - \frac{19(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{384m_1^3m_2^3} - \frac{5(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{384m_1^3m_2^3} + \frac{11(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{192m_1^3m_2^3} + \frac{77(\mathbf{p}_1\cdot\mathbf{p}_2)^3}{96m_1^3m_2^3} + \frac{233(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_2^2}{96m_1^3m_2^3} \\
&- \frac{47(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2\mathbf{p}_2^2}{32m_1^3m_2^3} + \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{384m_1^3m_2^3} - \frac{185\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{384m_1^3m_2^3} - \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^4}{4m_1^2m_2^4} + \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_2)^4\mathbf{p}_1^2}{4m_1^2m_2^4} - \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3(\mathbf{p}_1\cdot\mathbf{p}_2)}{2m_1^2m_2^4} + \frac{21(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{16m_1^2m_2^4} + \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_2^2}{6m_1^2m_2^4} \\
&+ \frac{49(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2\mathbf{p}_2^2}{48m_1^2m_2^4} - \frac{133(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{24m_1^2m_2^4} - \frac{77(\mathbf{p}_1\cdot\mathbf{p}_2)^2\mathbf{p}_2^2}{96m_1^2m_2^4} + \frac{197(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_2^2)^2}{96m_1^2m_2^4} - \frac{173\mathbf{p}_1^2(\mathbf{p}_2^2)^2}{48m_1^2m_2^4} + \frac{13(\mathbf{p}_2^2)^3}{8m_2^6}\Bigg) + \frac{G^3m_1m_2}{r_{12}^3}\Bigg(m_1^2\bigg(\frac{5027(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4}{384m_1^4} - \frac{22993(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2}{960m_1^4} - \frac{6695(\mathbf{p}_1^2)^2}{1152m_1^4} \\
&- \frac{3191(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{640m_1^3m_2} + \frac{28561(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2}{1920m_1^3m_2} + \frac{8777(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{384m_1^3m_2} + \frac{752969\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{28800m_1^3m_2} - \frac{16481(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{960m_1^2m_2^2} + \frac{94433(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2}{4800m_1^2m_2^2} - \frac{103957(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)}{2400m_1^2m_2^2} \\
&+ \frac{791(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{400m_1^2m_2^2} + \frac{26627(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_2^2}{1600m_1^2m_2^2} - \frac{118261\mathbf{p}_1^2\mathbf{p}_2^2}{4800m_1^2m_2^2} + \frac{105(\mathbf{p}_2^2)^2}{32m_2^4}\bigg) + m_1m_2\bigg(\left(\frac{2749\pi^2}{8192} - \frac{211189}{19200}\right)\frac{(\mathbf{p}_1^2)^2}{m_1^4} + \left(\frac{63347}{1600} - \frac{1059\pi^2}{1024}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2}{m_1^4} + \left(\frac{375\pi^2}{8192} - \frac{23533}{1280}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4}{m_1^4} \\
&+ \left(\frac{10631\pi^2}{8192} - \frac{1918349}{57600}\right)\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{m_1^2m_2^2} + \left(\frac{13723\pi^2}{16384} - \frac{2492417}{57600}\right)\frac{\mathbf{p}_1^2\mathbf{p}_2^2}{m_1^2m_2^2} + \left(\frac{1411429}{19200} - \frac{1059\pi^2}{512}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2}{m_1^2m_2^2} + \left(\frac{248991}{6400} - \frac{6153\pi^2}{2048}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^2m_2^2} \\
&- \left(\frac{30383}{960} + \frac{36405\pi^2}{16384}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2m_2^2} + \left(\frac{1243717}{14400} - \frac{40483\pi^2}{16384}\right)\frac{\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^3m_2} + \left(\frac{2369}{60} + \frac{35655\pi^2}{16384}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^3m_2} + \left(\frac{43101\pi^2}{16384} - \frac{391711}{6400}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2}{m_1^3m_2} \\
&+ \left(\frac{56955\pi^2}{16384} - \frac{1646983}{19200}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^3m_2}\bigg)\Bigg) + \frac{G^4m_1m_2}{r_{12}^4}\Bigg(m_1^3\bigg(\frac{64861\mathbf{p}_1^2}{4800m_1^2} - \frac{91(\mathbf{p}_1\cdot\mathbf{p}_2)}{8m_1m_2} + \frac{105\mathbf{p}_2^2}{32m_2^2} - \frac{9841(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2}{1600m_1^2} - \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{2m_1m_2}\bigg) + m_1^2m_2\bigg(\left(\frac{1937033}{57600} - \frac{199177\pi^2}{49152}\right)\frac{\mathbf{p}_1^2}{m_1^2} \\
&+ \left(\frac{176033\pi^2}{24576} - \frac{2864917}{57600}\right)\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1m_2} + \left(\frac{282361}{19200} - \frac{21837\pi^2}{8192}\right)\frac{\mathbf{p}_2^2}{m_2^2} + \left(\frac{698723}{19200} + \frac{21745\pi^2}{16384}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2}{m_1^2} + \left(\frac{63641\pi^2}{24576} - \frac{2712013}{19200}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1m_2} + \left(\frac{3200179}{57600} - \frac{28691\pi^2}{24576}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_2^2}\bigg)\Bigg) \\
&+ \frac{G^5m_1m_2}{r_{12}^5}\Bigg(-\frac{m_1^4}{16} + \left(\frac{6237\pi^2}{1024} - \frac{169799}{2400}\right)m_1^3m_2 + \left(\frac{44825\pi^2}{6144} - \frac{609427}{7200}\right)m_1^2m_2^2\Bigg) + (1 \leftrightarrow 2).
\end{aligned}
$$

- General-relativistic isolated systems admit the Poincaré group as a *global* symmetry. Therefore any coordinate system respecting asymptotic flatness should embody some representation of this symmetry.
- The presence of a Poincaré symmetry is equivalent to requiring the existence of generators $P^\mu(\mathbf{x}_a, \mathbf{p}_a)$ and $J^{\mu\nu}(\mathbf{x}_a, \mathbf{p}_a, t)$ realized as functions on the extended two-body phase-space $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{p}_1, \mathbf{p}_2, t)$, whose usual Poisson brackets

$$\{f(\mathbf{x}_a, \mathbf{p}_a, t), g(\mathbf{x}_a, \mathbf{p}_a, t)\} := \sum_a \left(\frac{\partial f}{\partial x_a^i} \frac{\partial g}{\partial p_{ai}} - \frac{\partial f}{\partial p_{ai}} \frac{\partial g}{\partial x_a^i} \right)$$

satisfy the Poincaré algebra [here $\eta_{\mu\nu} = \mathrm{diag}\,(-1, +1, +1, +1)$ and $c = 1$]:

$$\{P^\mu, P^\nu\} = 0,$$

$$\{P^\mu, J^{\rho\sigma}\} = -\eta^{\mu\rho}\, P^\sigma + \eta^{\mu\sigma}\, P^\rho,$$

$$\{J^{\mu\nu}, J^{\rho\sigma}\} = -\eta^{\nu\rho}\, J^{\mu\sigma} + \eta^{\mu\rho}\, J^{\nu\sigma} + \eta^{\sigma\mu}\, J^{\rho\nu} - \eta^{\sigma\nu}\, J^{\rho\mu}.$$

- One decomposes P^μ and $J^{\mu\nu}$ into: the total energy $P^0 = H$ (including the rest-mass contribution), three momentum P^i, angular momentum $J^i := \frac{1}{2}\,\varepsilon^{ik\ell}\, J_{k\ell}$, and boost vector $K^i := J^{i0}$ (K^i represents the constant of motion associated to the center of mass theorem).
- One further decomposes the boost vector K^i as

$$K^i(\mathbf{x}_a, \mathbf{p}_a, t) \equiv G^i(\mathbf{x}_a, \mathbf{p}_a) - t\, P^i(\mathbf{x}_a, \mathbf{p}_a),$$

where G^i is the center-of-mass(energy) vector.

Poincaré Algebra (2/4)

- The Poincaré algebra relations explicitly read

$$\{P_i\,, P_j\} = 0, \quad \{J_i,\, J_j\} = \epsilon_{ijk} J_k,$$

$$\{J_i,\, P_j\} = \epsilon_{ijk} P_k,$$

$$\{P_i,\, H\} = 0, \quad \{J_i,\, H\} = 0,$$

$$\{J_i,\, G_j\} = \epsilon_{ijk} G_k,$$

$$\{G_i,\, H\} = P_i,$$

$$\{G_i,\, P_j\} = c^{-2} H \delta_{ij},$$

$$\{G_i,\, G_j\} = -c^{-2} \epsilon_{ijk} J_k.$$

These relations have to be fulfilled with 4PN accuracy.

Poincaré Algebra (3/4)

- The Hamiltonian H entering Poincaré algebra is the full 4PN-accurate Hamiltonian,

$$H_{\leq 4\mathrm{PN}}[\mathbf{x}_a, \mathbf{p}_a] = H^{\mathrm{local}}_{\leq 4\mathrm{PN}}(\mathbf{x}_a, \mathbf{p}_a) + H^{\mathrm{nonlocal}}_{4\mathrm{PN}}[\mathbf{x}_a, \mathbf{p}_a],$$

$$H^{\mathrm{local}}_{\leq 4\mathrm{PN}}(\mathbf{x}_a, \mathbf{p}_a) = \sum_a m_a c^2 + H_{\mathrm{N}}(\mathbf{x}_a, \mathbf{p}_a) + H_{1\mathrm{PN}}(\mathbf{x}_a, \mathbf{p}_a) + H_{2\mathrm{PN}}(\mathbf{x}_a, \mathbf{p}_a) + H_{3\mathrm{PN}}(\mathbf{x}_a, \mathbf{p}_a) + H^{\mathrm{local}}_{4\mathrm{PN}}(\mathbf{x}_a, \mathbf{p}_a).$$

- The nonlocal-in-time piece $H^{\mathrm{nonlocal}}_{4\mathrm{PN}}$ is Galileo invariant, because

$$\dddot{I}_{ij} = -2\,\frac{G\mu\,M}{r_{12}^3}\left(4\,x_{12}^{\langle i}v_{12}^{j\rangle} - \frac{3}{r_{12}}\,(\mathbf{n}_{12}\cdot\mathbf{v}_{12})\,x_{12}^{\langle i}\,x_{12}^{j\rangle}\right).$$

Poincaré Algebra (4/4)

- The Hamiltonian H is translationally and rotationally invariant, therefore the total linear and angular momenta are simply realized as

$$P_i(\mathbf{x}_a, \mathbf{p}_a) = \sum_a p_{ai}, \quad J_i(\mathbf{x}_a, \mathbf{p}_a) = \sum_a \epsilon_{ijk} x_a^j p_{ak}.$$

- They exactly satisfy the relations $\{P_i, P_j\} = 0$, $\{J_i, J_j\} = \epsilon_{ijk} J_k$, $\{J_i, P_j\} = \epsilon_{ijk} P_k$, $\{P_i, H\} = 0$, $\{J_i, H\} = 0$.

Looking for the 4PN-Accurate Center-of-Mass Vector: the Method of Undetermined Coefficients (1/2)

- The condition for full Poincaré invariance boils down to the existence of a center-of-mass three-vector $\mathbf{G}$ satisfying the three non-trivial relations

$$\{G_i,\, H\} = P_i, \quad \{G_i,\, P_j\} = c^{-2} H \delta_{ij}, \quad \{G_i,\, G_j\} = -c^{-2} \epsilon_{ijk} J_k.$$

- One constructs $\mathbf{G}$ as a vector from $\mathbf{x}_a$ and $\mathbf{p}_a$ only, therefore the relation

$$\{J_i,\, G_j\} = \epsilon_{ijk} G_k$$

will be exactly satisfied.

- The generic form of the three-vector $\mathbf{G}$ reads

$$\mathbf{G}(\mathbf{x}_a, \mathbf{p}_a) = \sum_a \Big(M_a(\mathbf{x}_b, \mathbf{p}_b)\, \mathbf{x}_a + N_a(\mathbf{x}_b, \mathbf{p}_b)\, \mathbf{p}_a \Big),$$

where M_a and N_a possess the following 4PN-accurate expansions

$$M_a = m_a + M_a^{\mathrm{1PN}} + M_a^{\mathrm{2PN}} + M_a^{\mathrm{3PN}} + M_a^{\mathrm{4PN}},$$

$$N_a = N_a^{\mathrm{2PN}} + N_a^{\mathrm{3PN}} + N_a^{\mathrm{4PN}}.$$

Looking for the 4PN-Accurate Center-of-Mass Vector: the Method of Undetermined Coefficients (2/2)

- One writes the most general expressions for the successive PN approximations to the functions M_a and N_a as sums of scalar monomials of the form

$$c_{\mathbf{n}}\, r_{12}^{-n_0} \left(\frac{\mathbf{p}_1^2}{m_1^2}\right)^{n_1} \left(\frac{\mathbf{p}_1\cdot\mathbf{p}_2}{m_1 m_2}\right)^{n_2} \left(\frac{\mathbf{p}_2^2}{m_2^2}\right)^{n_3} \left(\frac{\mathbf{n}_{12}\cdot\mathbf{p}_1}{m_1}\right)^{n_4} \left(\frac{\mathbf{n}_{12}\cdot\mathbf{p}_2}{m_2}\right)^{n_5} m_1^{n_6}\, m_2^{n_7},$$

  where $n_0,\dots,n_7$ are nonnegative integers, $c_{\mathbf{n}}$ are *dimensionless* coefficients to be determined.

- One constrains the possible values of $n_0,\dots,n_7$ using dimensional analysis, Euclidean covariance (including parity symmetry), time reversal symmetry (which imposes that M_a is even and N_a is odd under the operation $\mathbf{p}_a \to -\mathbf{p}_a$), and the $1 \leftrightarrow 2$ relabeling symmetry.

- At the 4PN level the most general patterns for the functions M_a^{4PN} and N_a^{4PN} involve 210 coefficients $c_{\mathbf{n}}$.

- To find them it is enough to use only the relation

$$\{G_i,\, H\} = P_i.$$

  At the 4PN level it yields 525 equations to be satisfied by the coefficients $c_{\mathbf{n}}$.

- One finds a unique solution to these equations. Then one checks that this solution satisfies the remaining two Poincaré algebra relations:

$$\{G_i,\, P_j\} = c^{-2} H\delta_{ij}, \quad \{G_i,\, G_j\} = -c^{-2}\epsilon_{ijk}J_k.$$

4PN-Accurate Center-of-Mass Vector (Jaranowski-Schäfer 2012, Damour-Jaranowski-Schäfer 2014)

$$
\begin{aligned}
\mathbf{G}_{\le 4\mathrm{PN}}(\mathbf{x}_a,\mathbf{p}_a) &= m_1\mathbf{x}_1 + \frac{1}{c^2}\bigg(\frac{\mathbf{p}_1^2}{2m_1} - \frac{Gm_1m_2}{2r_{12}}\bigg)\mathbf{x}_1 + \frac{1}{c^4}\bigg\{\bigg(-\frac{1}{8}\frac{(\mathbf{p}_1^2)^2}{m_1^3} + \frac{1}{4}\frac{Gm_1m_2}{r_{12}}\bigg(-5\frac{\mathbf{p}_1^2}{m_1^2} - \frac{\mathbf{p}_2^2}{m_2^2} + 7\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1m_2} + \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1m_2}\bigg) + \frac{1}{4}\frac{Gm_1m_2}{r_{12}}\frac{G(m_1+m_2)}{r_{12}}\bigg)\mathbf{x}_1 - \frac{5}{4}G(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1\bigg\} + \frac{1}{c^6}\bigg\{\bigg[\frac{(\mathbf{p}_1^2)^3}{16m_1^5} \\
&\quad + \frac{1}{16}\frac{Gm_1m_2}{r_{12}}\bigg(9\frac{(\mathbf{p}_1^2)^2}{m_1^4} + \frac{(\mathbf{p}_2^2)^2}{m_2^4} - 11\frac{\mathbf{p}_1^2\mathbf{p}_2^2}{m_1^2m_2^2} - 2\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{m_1^2m_2^2} + 3\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2m_2^2} + 7\frac{\mathbf{p}_2^2(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2}{m_1^2m_2^2} - 12\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^2m_2^2} - 3\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2m_2^2}\bigg) + \frac{1}{24}\frac{G^2m_1m_2}{r_{12}^2}\bigg((112m_1+45m_2)\frac{\mathbf{p}_1^2}{m_1^2} + (15m_1+2m_2)\frac{\mathbf{p}_2^2}{m_2^2} \\
&\quad - \frac{1}{2}(209m_1+115m_2)\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1m_2} - (31m_1+5m_2)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1m_2} + \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2}{m_1} - \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_2}\bigg) - \frac{1}{8}\frac{Gm_1m_2}{r_{12}}\frac{G^2(m_1^2+5m_1m_2+m_2^2)}{r_{12}^2}\bigg]\mathbf{x}_1 + \bigg[\frac{1}{8}\frac{G}{m_1m_2}\big(2(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_2) - \mathbf{p}_2^2(\mathbf{n}_{12}\cdot\mathbf{p}_1) + 3(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\big) \\
&\quad + \frac{1}{48}\frac{G^2}{r_{12}}\big(19m_2(\mathbf{n}_{12}\cdot\mathbf{p}_1) + (130m_1+137m_2)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\big)\bigg]\mathbf{p}_1\bigg\} + \frac{1}{c^8}\bigg\{\bigg[-\frac{5(\mathbf{p}_1^2)^4}{128m_1^7} + \frac{Gm_1m_2}{r_{12}}\bigg(-\frac{13(\mathbf{p}_1^2)^3}{32m_1^6} - \frac{15(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{256m_1^4m_2^2} + \frac{45(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2}{128m_1^4m_2^2} - \frac{91(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1^2)^2}{256m_1^4m_2^2} - \frac{5(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)}{32m_1^4m_2^2} \\
&\quad + \frac{25(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{32m_1^4m_2^2} + \frac{5(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{64m_1^4m_2^2} + \frac{7\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{64m_1^4m_2^2} + \frac{11(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4\mathbf{p}_2^2}{256m_1^4m_2^2} - \frac{47(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2\mathbf{p}_2^2}{128m_1^4m_2^2} + \frac{91(\mathbf{p}_1^2)^2\mathbf{p}_2^2}{256m_1^4m_2^2} + \frac{5(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3}{32m_1^3m_2^3} - \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3\mathbf{p}_1^2}{32m_1^3m_2^3} + \frac{15(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{32m_1^3m_2^3} \\
&\quad + \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{32m_1^3m_2^3} - \frac{5(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{16m_1^3m_2^3} - \frac{(\mathbf{p}_1\cdot\mathbf{p}_2)^3}{16m_1^3m_2^3} - \frac{11(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_2^2}{32m_1^3m_2^3} + \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2\mathbf{p}_2^2}{32m_1^3m_2^3} - \frac{5(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{32m_1^3m_2^3} + \frac{\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{32m_1^3m_2^3} + \frac{15(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^4}{256m_1^2m_2^4} - \frac{11(\mathbf{n}_{12}\cdot\mathbf{p}_2)^4\mathbf{p}_1^2}{256m_1^2m_2^4} \\
&\quad + \frac{5(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3(\mathbf{p}_1\cdot\mathbf{p}_2)}{32m_1^2m_2^4} - \frac{5(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{64m_1^2m_2^4} - \frac{21(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_2^2}{128m_1^2m_2^4} + \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2\mathbf{p}_2^2}{128m_1^2m_2^4} - \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{32m_1^2m_2^4} + \frac{(\mathbf{p}_1\cdot\mathbf{p}_2)^2\mathbf{p}_2^2}{64m_1^2m_2^4} + \frac{11(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_2^2)^2}{256m_1^2m_2^4} + \frac{37\mathbf{p}_1^2(\mathbf{p}_2^2)^2}{256m_1^2m_2^4} - \frac{(\mathbf{p}_2^2)^3}{32m_2^6}\bigg) \\
&\quad + \frac{G^2m_1m_2}{r_{12}^2}\bigg(m_1\bigg(\frac{7711(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4}{3840m_1^4} - \frac{2689(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2}{3840m_1^4} + \frac{2683(\mathbf{p}_1^2)^2}{1920m_1^4} - \frac{67(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{30m_1^3m_2} + \frac{1621(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2}{1920m_1^3m_2} - \frac{411(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{1280m_1^3m_2} - \frac{25021\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{3840m_1^3m_2} + \frac{289(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{128m_1^2m_2^2} - \frac{259(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2}{128m_1^2m_2^2} \\
&\quad + \frac{689(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)}{192m_1^2m_2^2} + \frac{11(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{48m_1^2m_2^2} - \frac{147(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_2^2}{64m_1^2m_2^2} + \frac{283\mathbf{p}_1^2\mathbf{p}_2^2}{64m_1^2m_2^2} + \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3}{12m_1m_2^3} + \frac{49(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{48m_1m_2^3} - \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_2^2}{6m_1m_2^3} - \frac{7(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{48m_1m_2^3} - \frac{9(\mathbf{p}_2^2)^2}{32m_2^4}\bigg) + m_2\bigg(-\frac{45(\mathbf{p}_1^2)^2}{32m_1^4} + \frac{7\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{48m_1^3m_2} \\
&\quad + \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2}{6m_1^3m_2} - \frac{49(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{48m_1^3m_2} - \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{12m_1^3m_2} + \frac{7(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{24m_1^2m_2^2} + \frac{635\mathbf{p}_1^2\mathbf{p}_2^2}{192m_1^2m_2^2} - \frac{983(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_2^2}{384m_1^2m_2^2} + \frac{413(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{384m_1^2m_2^2} - \frac{331(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2}{192m_1^2m_2^2} + \frac{437(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)}{64m_1^2m_2^2} \\
&\quad + \frac{11(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3}{15m_1m_2^3} - \frac{1349(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{1280m_1m_2^3} - \frac{5221(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_2^2}{1920m_1m_2^3} - \frac{2579(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{3840m_1m_2^3} + \frac{6769(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_2^2}{3840m_2^4} - \frac{2563(\mathbf{p}_2^2)^2}{1920m_2^4} - \frac{2037(\mathbf{n}_{12}\cdot\mathbf{p}_2)^4}{1280m_2^4}\bigg)\bigg) + \frac{G^3m_1m_2}{r_{12}^3}\bigg(m_1^2\bigg(-\frac{179843\mathbf{p}_1^2}{14400m_1^2} + \frac{10223(\mathbf{p}_1\cdot\mathbf{p}_2)}{1200m_1m_2} - \frac{15\mathbf{p}_2^2}{16m_2^2} \\
&\quad + \frac{8881(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{2400m_1m_2} + \frac{17737(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2}{1600m_1^2}\bigg) + m_1m_2\bigg(\bigg(\frac{8225\pi^2}{16384} - \frac{12007}{1152}\bigg)\frac{\mathbf{p}_1^2}{m_1^2} + \bigg(\frac{143}{16} - \frac{\pi^2}{64}\bigg)\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1m_2} + \bigg(\frac{655}{1152} - \frac{7969\pi^2}{16384}\bigg)\frac{\mathbf{p}_2^2}{m_2^2} + \bigg(\frac{6963\pi^2}{16384} - \frac{40697}{3840}\bigg)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2}{m_1^2} + \bigg(\frac{119}{16} + \frac{3\pi^2}{64}\bigg)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1m_2} \\
&\quad + \bigg(\frac{30377}{3840} - \frac{7731\pi^2}{16384}\bigg)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_2^2}\bigg) + m_2^2\bigg(-\frac{35\mathbf{p}_1^2}{16m_1^2} + \frac{1327(\mathbf{p}_1\cdot\mathbf{p}_2)}{1200m_1m_2} + \frac{52343\mathbf{p}_2^2}{14400m_2^2} - \frac{2581(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{2400m_1m_2} - \frac{15737(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{1600m_2^2}\bigg)\bigg) + \frac{G^4m_1m_2}{r_{12}^4}\bigg(\frac{m_1^3}{16} + \bigg(\frac{3371\pi^2}{6144} - \frac{6701}{1440}\bigg)m_1^2m_2 + \bigg(\frac{20321}{1440} - \frac{7403\pi^2}{6144}\bigg)m_1m_2^2 \\
&\quad + \frac{m_2^3}{16}\bigg)\bigg]\mathbf{x}_1 + \bigg[Gm_2\bigg(-\frac{5(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{64m_1^3m_2^2} + \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2}{64m_1^3m_2^2} + \frac{5(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)}{32m_1^3m_2^2} - \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{32m_1^3m_2^2} + \frac{3(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{32m_1^3m_2^2} - \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3\mathbf{p}_2^2}{64m_1^3m_2^2} - \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)\mathbf{p}_1^2\mathbf{p}_2^2}{64m_1^3m_2^2} + \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3}{32m_1^2m_2^3} \\
&\quad - \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3\mathbf{p}_1^2}{32m_1^2m_2^3} + \frac{3(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{16m_1^2m_2^3} + \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{16m_1^2m_2^3} - \frac{9(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_2^2}{32m_1^2m_2^3} + \frac{5(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2\mathbf{p}_2^2}{32m_1^2m_2^3} - \frac{3(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{16m_1^2m_2^3} - \frac{11(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^4}{128m_1m_2^4} + \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3(\mathbf{p}_1\cdot\mathbf{p}_2)}{32m_1m_2^4} + \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_2^2}{64m_1m_2^4} \\
&\quad + \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{32m_1m_2^4} - \frac{3(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{p}_2^2)^2}{128m_1m_2^4}\bigg) + \frac{G^2m_2}{r_{12}}\bigg(m_1\bigg(-\frac{387(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3}{1280m_1^3} + \frac{10429(\mathbf{n}_{12}\cdot\mathbf{p}_1)\mathbf{p}_1^2}{3840m_1^3} - \frac{751(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{480m_1^2m_2} + \frac{2209(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2}{640m_1^2m_2} - \frac{6851(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{p}_1\cdot\mathbf{p}_2)}{1920m_1^2m_2} + \frac{43(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{192m_1m_2^2} \\
&\quad - \frac{125(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)}{192m_1m_2^2} + \frac{25(\mathbf{n}_{12}\cdot\mathbf{p}_1)\mathbf{p}_2^2}{48m_1m_2^2} - \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3}{8m_2^3} + \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_2^2}{12m_2^3}\bigg) + m_2\bigg(\frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2}{48m_1^2m_2} + \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{p}_1\cdot\mathbf{p}_2)}{24m_1^2m_2} - \frac{49(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{48m_1^2m_2} + \frac{295(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{384m_1m_2^2} - \frac{5(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)}{24m_1m_2^2} - \frac{155(\mathbf{n}_{12}\cdot\mathbf{p}_1)\mathbf{p}_2^2}{384m_1m_2^2} \\
&\quad - \frac{5999(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3}{3840m_2^3} + \frac{11251(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_2^2}{3840m_2^3}\bigg)\bigg) + \frac{G^3m_2}{r_{12}^2}\bigg(m_1^2\bigg(-\frac{37397(\mathbf{n}_{12}\cdot\mathbf{p}_1)}{7200m_1} - \frac{12311(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{2400m_2}\bigg) + m_1m_2\bigg(\bigg(\frac{5005\pi^2}{8192} - \frac{81643}{11520}\bigg)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)}{m_1} + \bigg(\frac{773\pi^2}{2048} - \frac{61177}{11520}\bigg)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_2}\bigg) + m_2^2\bigg(-\frac{7073(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{1200m_2}\bigg)\bigg)\bigg]\mathbf{p}_1\bigg\} \\
&\quad + (1\leftrightarrow 2).
\end{aligned}
$$

Relative motion in the center-of-mass frame ($\mathbf{p}_1 + \mathbf{p}_2 = 0$):

$$\big(\mathbf{x}_1, \mathbf{x}_2, \mathbf{p}_1, \mathbf{p}_2\big) \to (\mathbf{r}, \mathbf{p});$$

reduced variables in the center-of-mass frame:

$$\mathbf{r} := \frac{\mathbf{x}_1 - \mathbf{x}_2}{GM}, \qquad \mathbf{p} := \frac{\mathbf{p}_1}{\mu} = -\frac{\mathbf{p}_2}{\mu}, \qquad \widehat{t} := \frac{t}{GM}.$$

'Non-relativistic' (i.e. without rest-mass contribution) ADM Hamiltonian of the 2-point-mass system:

$$\widehat{H}^{\mathrm{NR}} := \frac{H - (m_1 + m_2)c^2}{\mu}.$$

Conservative 3PN-accurate Hamiltonian describing relative motion

$$\widehat{H}^{\mathrm{NR}}(\mathbf{r},\mathbf{p}) = \widehat{H}_{\mathrm{N}}(\mathbf{r},\mathbf{p}) + \frac{1}{c^2}\widehat{H}_{1\mathrm{PN}}(\mathbf{r},\mathbf{p}) + \frac{1}{c^4}\widehat{H}_{2\mathrm{PN}}(\mathbf{r},\mathbf{p}) + \frac{1}{c^6}\widehat{H}_{3\mathrm{PN}}(\mathbf{r},\mathbf{p}),$$

$$\widehat{H}_{\mathrm{N}}(\mathbf{r},\mathbf{p}) = \frac{1}{2}\,\mathbf{p}^2 - \frac{1}{r}, \quad \ldots .$$

The functional form of the Hamiltonian $\widehat{H}^{\mathrm{NR}}(\mathbf{r},\mathbf{p})$ is gauge dependent.

The relative-motion Hamiltonian $\widehat{H}^{\mathsf{NR}}$ is invariant under time translations and spatial rotations:

$$\widehat{H}^{\mathsf{NR}}(\mathbf{r},\mathbf{p}) = h(r,(\mathbf{n}\cdot\mathbf{p})^2,\mathbf{p}^2) \quad (\mathbf{n} := \mathbf{r}/r),$$

therefore energy and angular momentum of the system are conserved:

$$\widehat{E} := \widehat{H}^{\mathsf{NR}}(\mathbf{r},\mathbf{p}), \qquad \mathbf{j} := \frac{\mathbf{J}}{G\mu M} = \mathbf{r}\times\mathbf{p}$$

$$\big(\mathbf{J} := \mathbf{x}_1\times\mathbf{p}_1 + \mathbf{x}_2\times\mathbf{p}_2\big).$$

Polar coordinates (r, ϕ) in the plane of the relative trajectory:

$$\mathbf{r} = (r\cos\phi, r\sin\phi, 0).$$

The Hamilton-Jacobi equation

$$\frac{\partial \widehat{S}}{\partial \widehat{t}} + \widehat{H}^{\mathrm{NR}}\left(\mathbf{r}, \frac{\partial \widehat{S}}{\partial \mathbf{r}}\right) = 0$$

has solution for the action integral $\widehat{S}$ of the form

$$\widehat{S} := \frac{S}{G\mu M} = -\widehat{E}\,\widehat{t} + j\,\phi + \int \mathrm{d}r \sqrt{R(r;\widehat{E},j)},$$

$R(r;\widehat{E},j)$ is the effective radial potential.

R is obtained by solving the equation

$$\widehat{E} = \widehat{H}^{\rm NR}(\mathbf{r}, \mathbf{p}) = h(r, (\mathbf{n}\cdot\mathbf{p})^2, \mathbf{p}^2)$$

for p_r^2 [where $p_r := (\mathbf{n}\cdot\mathbf{p})$], after having replaced $\mathbf{p}^2$ by

$$\mathbf{p}^2 = (\mathbf{n}\cdot\mathbf{p})^2 + (\mathbf{n}\times\mathbf{p})^2 = p_r^2 + \frac{j^2}{r^2}, \quad \text{with} \quad j := |\mathbf{j}| :$$

$$\widehat{E} = h(r, p_r^2, p_r^2 + j^2/r^2) \implies R = R(r; \widehat{E}, j) = p_r^2(r; \widehat{E}, j).$$

The effective radial potential $R(r; \widehat{E}, j)$ is given, at the 3PN order, by the 7th-order polynomial in $1/r$:

$$R(r; \widehat{E}, j) = A + \frac{2B}{r} + \frac{C}{r^2} + \frac{D_1^{\rm 1PN}}{r^3} + \frac{D_2^{\rm 2PN}}{r^4} + \frac{D_3^{\rm 2PN}}{r^5} + \frac{D_4^{\rm 3PN}}{r^6} + \frac{D_5^{\rm 3PN}}{r^7}.$$

The coefficients A, B, C start at Newtonian order, while the extra terms $D_i^{n\rm PN}/r^{i+2}$ start at the indicated PN order. All the coefficients are polynomials in E and j^2.

The Hamilton-Jacobi theory shows that the observables of the motion are deducible from the (reduced) radial action integral,

$$i_r(\widehat{E}, j) := \frac{1}{\pi} \int_{r_{\rm min}}^{r_{\rm max}} \mathrm{d}r \sqrt{R(r, \widehat{E}, j)} = -j + \frac{1}{\sqrt{-\widehat{E}}} + \mathcal{O}(c^{-2}).$$

Periastron-to-periastron period:

$$\frac{P}{2\pi GM} = \frac{\partial}{\partial \widehat{E}} i_r(\widehat{E}, j) = \frac{1}{2}(-\widehat{E})^{-3/2} + \mathcal{O}(c^{-2}),$$

periastron advance per orbit ($\Delta\Phi := \Phi - 2\pi$, where Φ is the change of the angle ϕ between the two consecutive periastrons):

$$\frac{\Delta\Phi}{2\pi} = 1 - \frac{\partial}{\partial j} i_r(\widehat{E}, j) = 0 + \mathcal{O}(c^{-2}).$$

Delaunay variables (n, j, m):

$$n := i_r + j = \frac{\mathcal{N}}{G\mu M}, \quad j := \frac{J}{G\mu M}, \quad m := j_z = \frac{J_z}{G\mu M},$$

in the quantum language,
$\mathcal{N}/\hbar$ is the principal quantum number,
$J/\hbar$ is the total angular momentum quantum number,
$J_z/\hbar$ is the magnetic quantum number (by rotational symmetry, the magnetic quantum number does not enter the Hamiltonian);
they are adiabatic invariants of the dynamics,
and they are (approximately) quantized in integers.

$$n - j = i_r(\widehat{E}, j) \quad \Longrightarrow \quad \widehat{E} = \widehat{H}(n, j).$$

3PN-accurate 2-point-mass Delaunay Hamiltonian

$$\widehat{H}(n,j) = -\frac{1}{2n^2}\left(1 + \frac{1}{c^2}h_{1\mathrm{PN}}(n,j) + \frac{1}{c^4}h_{2\mathrm{PN}}(n,j) + \frac{1}{c^6}h_{3\mathrm{PN}}(n,j)\right),$$

$$h_{1\mathrm{PN}}(n,j) = \frac{6}{jn} - \frac{1}{4}(15-\nu)\frac{1}{n^2},$$

$$h_{2\mathrm{PN}}(n,j) = \frac{5}{2}(7-2\nu)\frac{1}{j^3 n} + \frac{27}{j^2 n^2} - \frac{3}{2}(35-4\nu)\frac{1}{jn^3} + \frac{1}{8}(145-15\nu+\nu^2)\frac{1}{n^4},$$

$$\begin{aligned} h_{3\mathrm{PN}}(n,j) = {} & \frac{1}{64}\left(7392 + (123\pi^2 - 8000)\nu + 336\nu^2\right)\frac{1}{j^5 n} + \frac{45}{2}(7-2\nu)\frac{1}{j^4 n^2} \\ & - \frac{1}{192}\left(14544 + (123\pi^2 - 22832)\nu + 1920\nu^2\right)\frac{1}{j^3 n^3} - \frac{45}{2}(20-3\nu)\frac{1}{j^2 n^4} \\ & + \frac{3}{2}(275 - 50\nu + 4\nu^2)\frac{1}{jn^5} - \frac{1}{64}(6363 - 805\nu + 90\nu^2 - 5\nu^3)\frac{1}{n^6}. \end{aligned}$$

Angular frequencies of the rosette motion:

$$\omega_{\text{radial}} = \frac{2\pi}{P} = \frac{1}{GM}\frac{\partial \widehat{H}(n,j)}{\partial n} = \frac{1}{GMn^3} + \mathcal{O}(c^{-2}),$$

$$\omega_{\text{periastron}} = \frac{\Delta\Phi}{P} = \frac{1}{GM}\frac{\partial \widehat{H}(n,j)}{\partial j} = 0 + \mathcal{O}(c^{-2}).$$

Open Issues (1/2)

- Completion of computations of 5PN, 5.5PN, 6PN, ... EOM of two-point-mass systems together with computation of gravitational-wave luminosities at 5PN, 5.5PN, 6PN, ... orders, and construction of $\geq$5PN-accurate templates for inspiralling compact binaries.
- Computation, within the PN framework, higher-order spin-dependent effects and, in the case of binaries containing neutron stars, higher-order tidal corrections.
- Higher-order perturbative solutions of two-body problem are complicated, both from computational and from conceptual point of view. Therefore it is highly desired to have more than one independent derivation of any analytical result:
  - — making independent derivations (e.g. within the ADM Hamiltonian approach) of gravitational-wave luminosities of two-point-mass system at the 2PN, 3PN, 4PN, ... order;
  - — making rederivation of 4PN two-body equations of motion using an extended body model (in $d = 3$ space dimensions).

Open Issues (2/2)

- Looking for a new treatment of regularization issues related to usage of δ-sources, which would simplify higher-order PN computations.
  - — Replacing δ-sources by sources described by some sequence of classical functions ("the δ sequence"); then there is no need for using distributional derivatives of singular homogeneous functions.
    At the 3PN level DJS (2008) successfully recomputed all UV logarithmically divergent terms using d-dimensional Riesz kernels to model point masses.
  - — Looking for some extension/modification of Schwartz distribution theory that would be suitable for purely 3-dimensional regularization.
    Such an attempt was made by Blanchet & Faye (2000), but their "extended Hadamard regularization" can not be combined with DR.
- Recompute and regularize IR divergences in the 4PN two-point-mass ADM Hamiltonian, without usage of gravitational self-force results and without introducing any ambiguity parameter.
- Try to increase the level of algorithmization and automatization of computation of 2-point-mass ADM Hamiltonians,
  starting from PN iterations of constraint equations up to UV/IR regularization, performed using a mixture of 3-dimensional RH regularization and DR. Then compute 2-point-mass Hamiltonians at orders 4.5PN, 5PN, 5.5PN, ...

Exercises

1. Check all calculations in Example from Section 3 of these lectures.
2. Find $\phi_{(2)}$, $S_{(4)1}$, $S_{(4)2}$, and $\phi_{(4)}$ in d dimensions.
3. Find $V^i_{(3)}$ in d dimensions.
4. Compute the Newtonian Hamiltonian H_{N} in d dimensions, $H_{\mathrm{N}} = -\int \mathrm{d}^d\mathbf{x}\,\Delta\phi_{(4)}$.
5. Compute the 1PN Hamiltonian $H_{1\mathrm{PN}}$ in d dimensions, $H_{1\mathrm{PN}} = -\int \mathrm{d}^d\mathbf{x}\,\Delta\phi_{(6)}$. Hints. (i) Knowing that $h^{\mathrm{TT}}_{(4)ij} \sim r^{2-d}$ for $r \to \infty$, show that the term $\phi_{(2),ij}h^{\mathrm{TT}}_{(4)ij}$ does not contribute to the Hamiltonian. (ii) Show that $(\tilde{\pi}^{ij}_{(3)})^2 = 2\partial_i(V^i_{(3)}\tilde{\pi}^{ij}_{(3)}) - 2V^i_{(3)}\partial_j\tilde{\pi}^{ij}_{(3)}$. Does the term $\partial_i(V^i_{(3)}\tilde{\pi}^{ij}_{(3)})$ contribute to the Hamiltonian?
6. Using explicit formula for the leading-order dissipative Hamiltonian $H^{\mathrm{diss}}_{2.5\mathrm{PN}}(\mathbf{x}_a, \mathbf{p}_a, t)$ given in these lectures, compute instantaneous GW luminosity in the leading-order. Then, assuming that the bodies in the system are moving along circular orbits, average the instantaneous GW luminosity over one orbital period using Newtonian equations of motion. Computation perform in $d = 3$ space dimensions. Answer: $\mathcal{L}(x;\nu) = \frac{32c^5}{5G}\nu^2x^5$.

Answers to Exercises 2–5.

$$\kappa := \frac{\Gamma(d/2-1)}{4\pi^{d/2}},\ \phi_{(2)} = \kappa\sum_a m_a r_a^{2-d},\quad \phi_{(4)} = -\frac{1}{2}S_{(4)1} + \frac{d-2}{4(d-1)}S_{(4)2},\quad S_{(4)1} = -\kappa\sum_a \frac{\mathrm{p}_a^2}{m_a}r_a^{2-d},$$

$$S_{(4)2} = -\kappa^2\sum_a\sum_{b\neq a} m_a m_b r_{ab}^{2-d}r_a^{2-d},\quad V^i_{(3)} = \frac{\kappa}{8(d-1)}\sum_a\Big((3d-2)p_{ai} + (d-2)^2(\mathrm{n}_a\cdot\mathrm{p}_a)n_a^i\Big)r_a^{2-d},$$

$$H_{\mathrm{N}}(\mathbf{x}_a,\mathbf{p}_a) = \frac{\mathbf{p}_1^2}{2m_1} - \frac{\kappa(d-2)}{4(d-1)}\frac{m_1m_2}{r_{12}^{d-2}} + (1\leftrightarrow 2),$$

$$H_{1\mathrm{PN}}(\mathbf{x}_a,\mathbf{p}_a) = -\frac{(\mathbf{p}_1^2)^2}{8m_1^3} + \frac{\kappa}{4(d-1)}\Big(\frac{1}{2}(3d-2)(\mathbf{p}_1\cdot\mathbf{p}_2) - d\,\frac{m_2}{m_1}\mathbf{p}_1^2 + \frac{1}{2}(d-2)^2(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\Big)\frac{1}{r_{12}^{d-2}}$$

$$+\frac{\kappa^2(d-2)^2}{8(d-1)^2}\frac{m_1^2m_2}{r_{12}^{2d-4}} + (1\leftrightarrow 2).$$

Equations Needed to Solve Exercises

$$\Delta\phi_{(2)} = -\sum_a m_a\delta_a, \quad \Delta\phi_{(4)} = \sum_a \Big(-\frac{\mathrm{p}_a^2}{2m_a} + \frac{(d-2)m_a}{4(d-1)}\,\phi_{(2)}\Big)\delta_a,$$

$$\Delta\phi_{(6)} = \sum_a \left\{ \frac{(\mathrm{p}_a^2)^2}{8m_a^3} + \frac{(d+2)\mathrm{p}_a^2}{8(d-1)m_a}\phi_{(2)} - \frac{(d-2)m_a}{8(d-1)}\Big(S_{(4)1} + \frac{d-2}{2(d-1)}(\phi_{(2)}^2 - S_{(4)2})\Big)\right\}\delta_a$$
$$- (\tilde{\pi}^{ij}_{(3)})^2 + \frac{d-2}{d-1}\phi_{(2),ij}h^{\mathrm{TT}}_{(4)ij},$$

$$\Delta V^i_{(3)} + \frac{d-2}{d}\partial_{ij}V^j_{(3)} = -\frac{1}{2}\sum_a p_{ai}\delta_a, \quad \tilde{\pi}^{ij}_{(3)} = \partial_i V^j_{(3)} + \partial_j V^i_{(3)} - \frac{2}{d}\delta^{ij}\partial_k V^k_{(3)},$$

$$\Delta S_{(4)1} = \sum_a \frac{\mathrm{p}_a^2}{m_a}\delta_a, \quad \Delta S_{(4)2} = \phi_{(2)}\sum_a m_a\delta_a.$$

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