GRAND collaboration meeting —Warsaw 2025

Radio Cherenkov effect

An analytical computation of the Cherenkov angles

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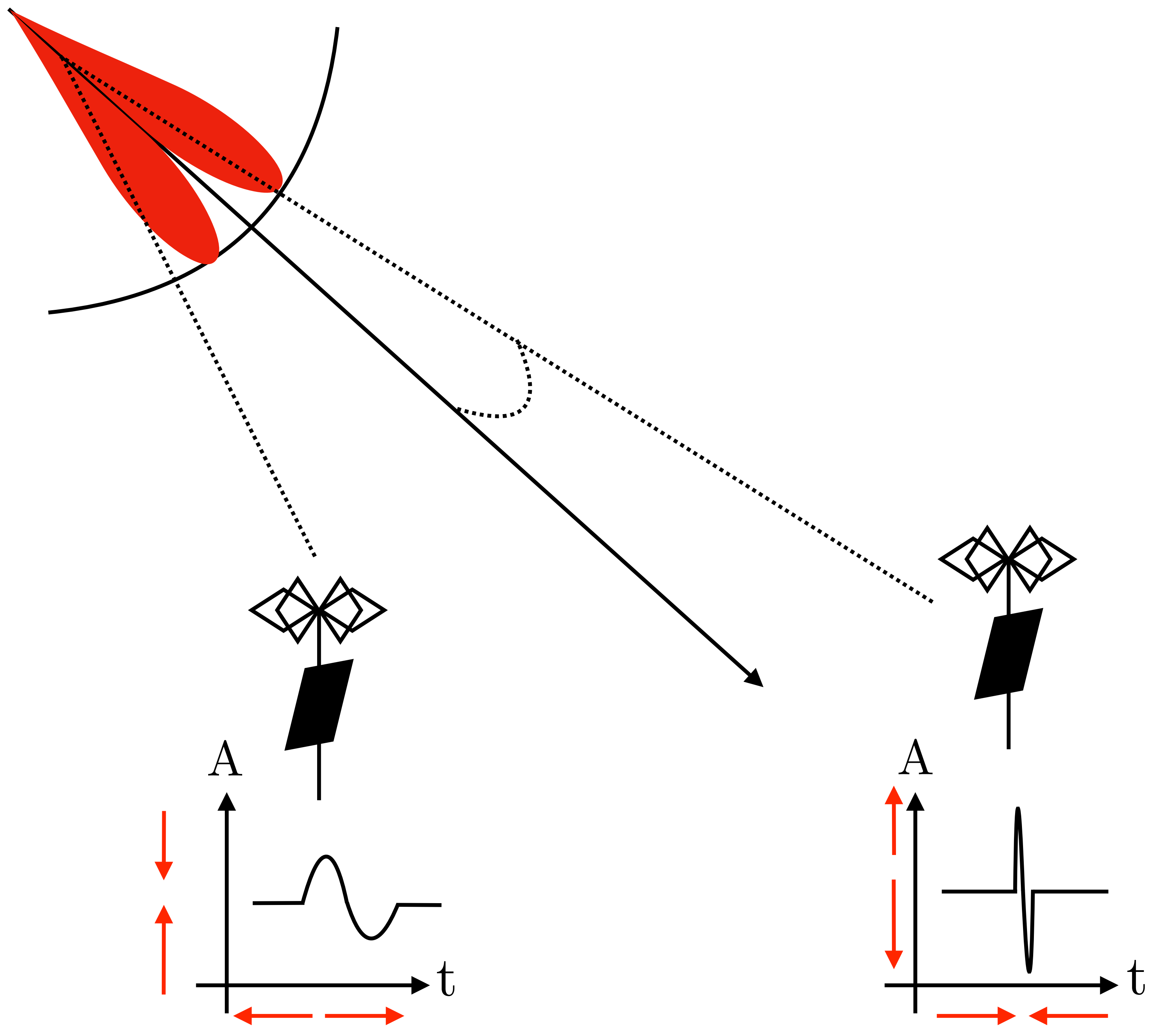
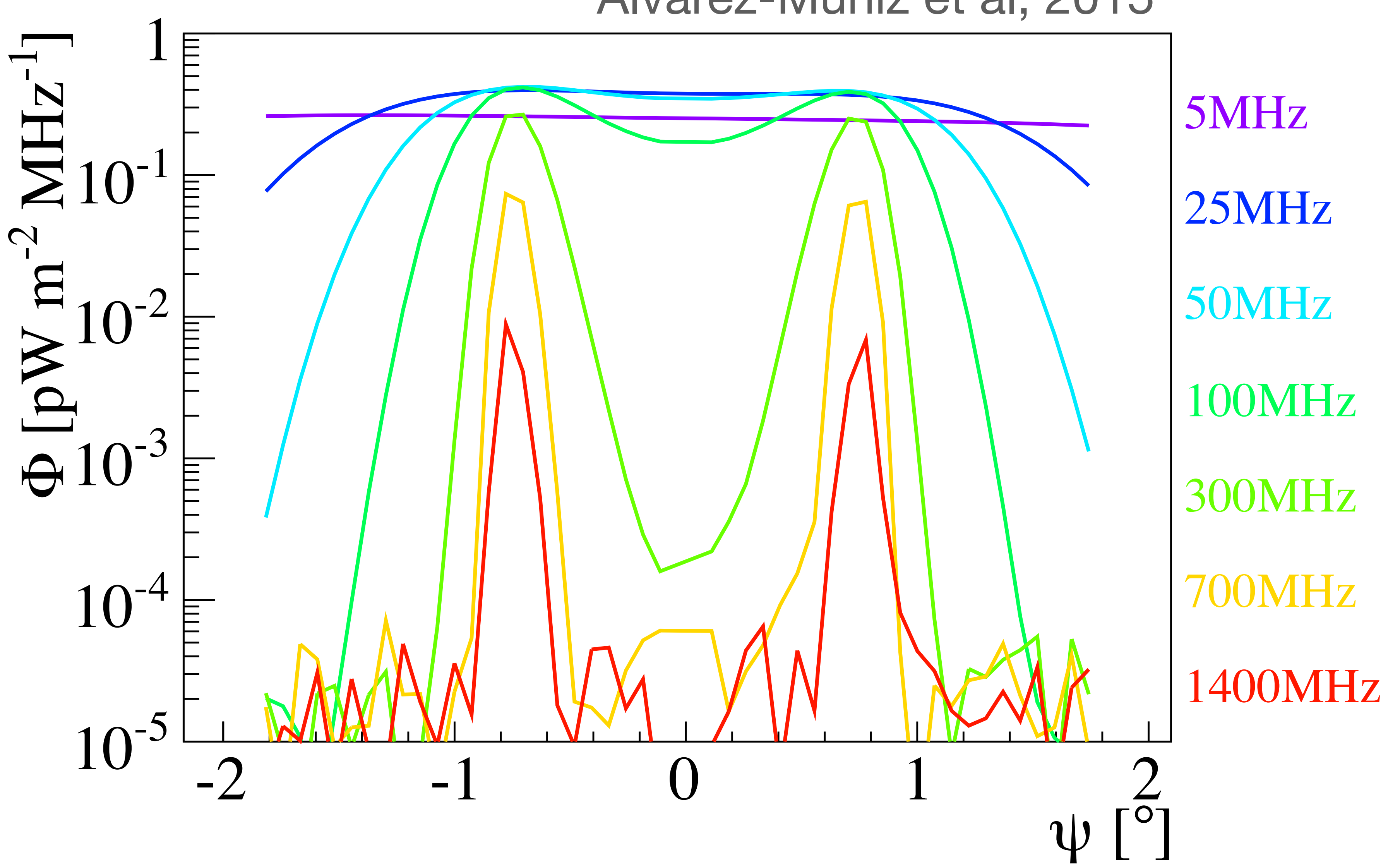
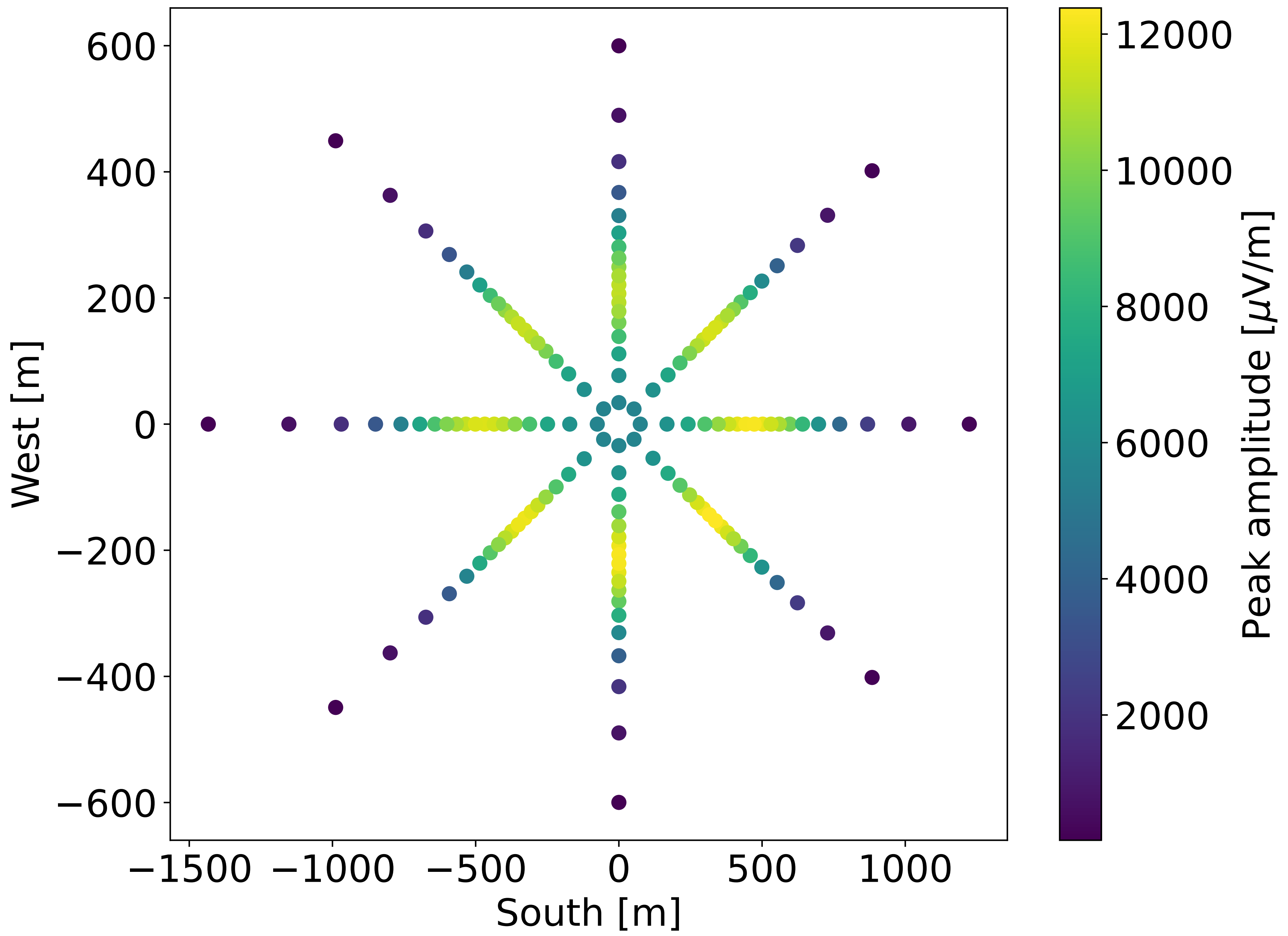
Radio Cherenkov effect

- EAS signals are amplified along specific directions
- Phenomena related to a time compression effect (propagation effect)



Alvarez-Muñiz et al, 2015

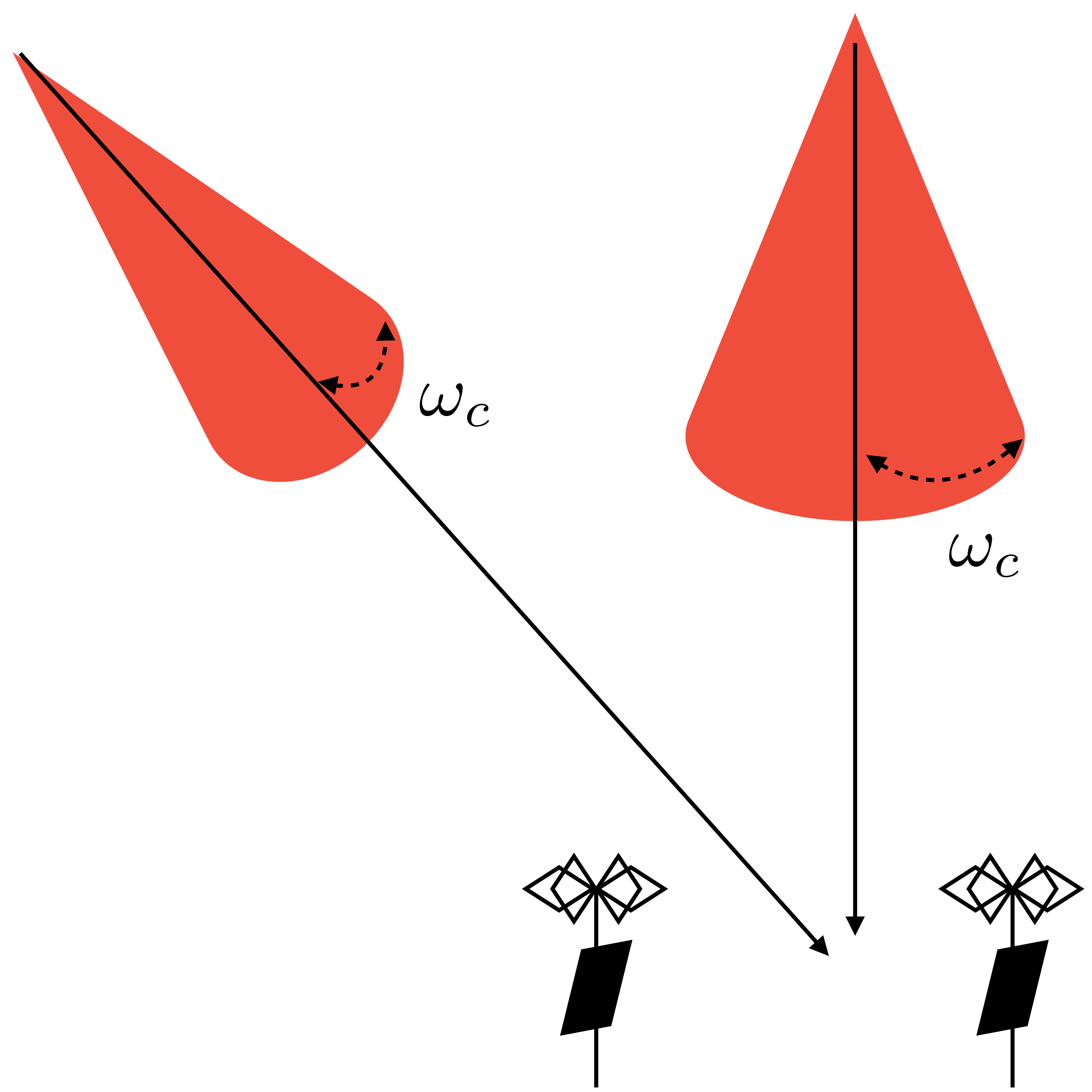


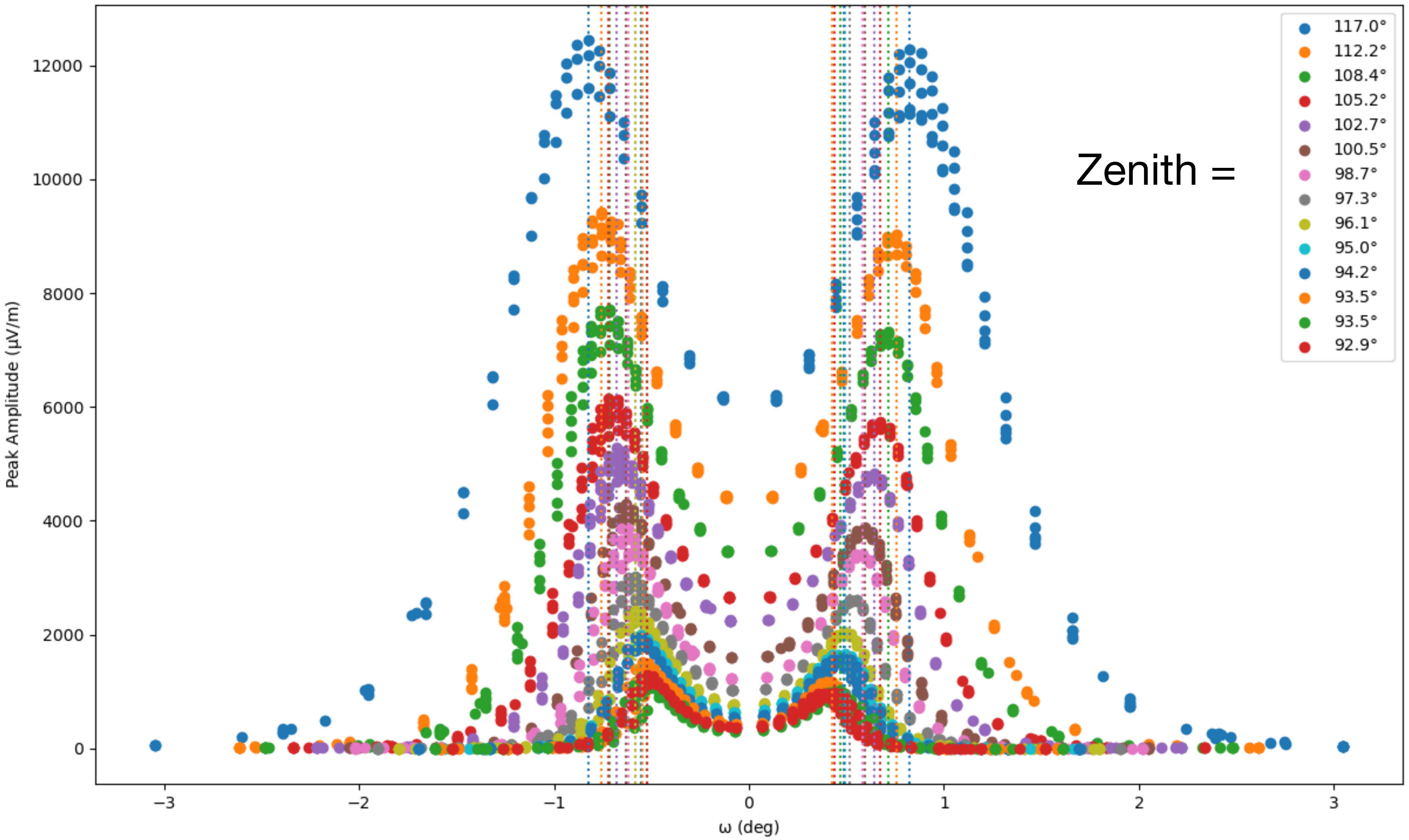


→ Cherenkov angle / Cherenkov cone

Cherenkov angle

- Time compression effect changes with shower configuration
- Seen through the change in the Cherenkov angle value





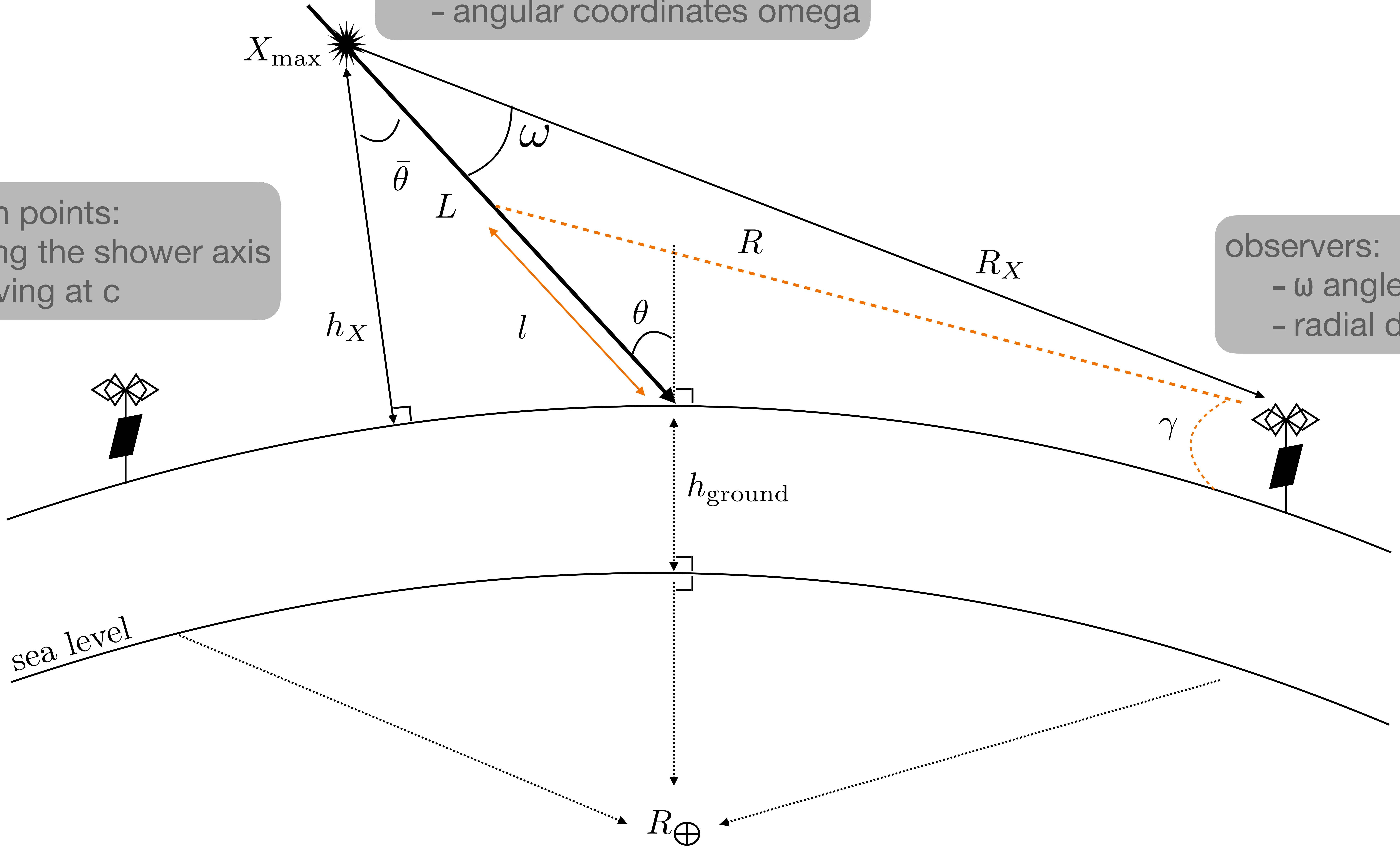
→ How to quantify and predict these changes in the Cherenkov angle value?

A bit of geometry

frame centered on Xmax:
- emission time (t') = 0
- angular coordinates omega

emission points:
- along the shower axis
- moving at c

observers:
- ω angle
- radial distance R



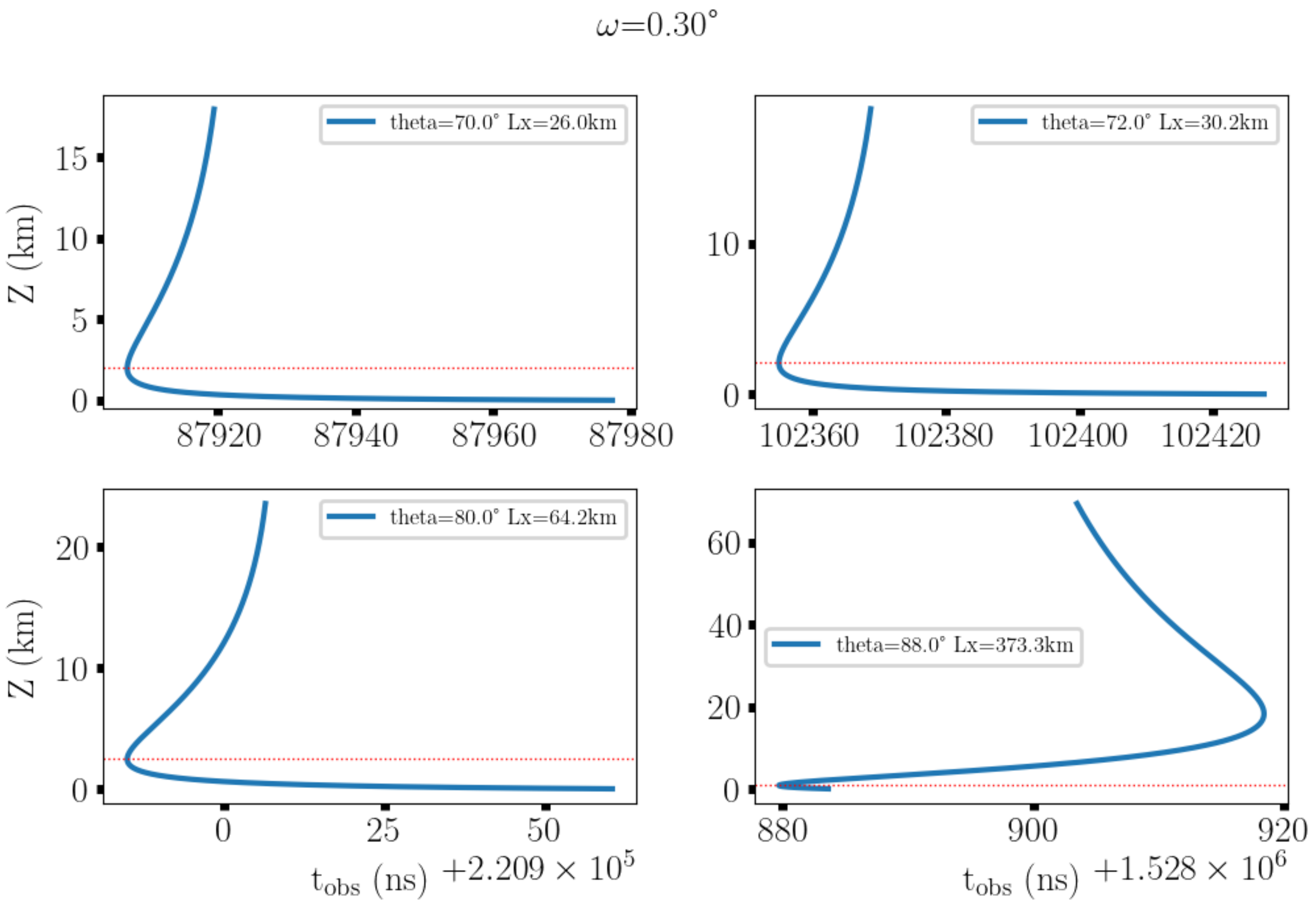
Retarded and observer times

- Causal relation: $c(t - t') = \langle n(R) \rangle \, R$

  + varying atmospheric refractive index

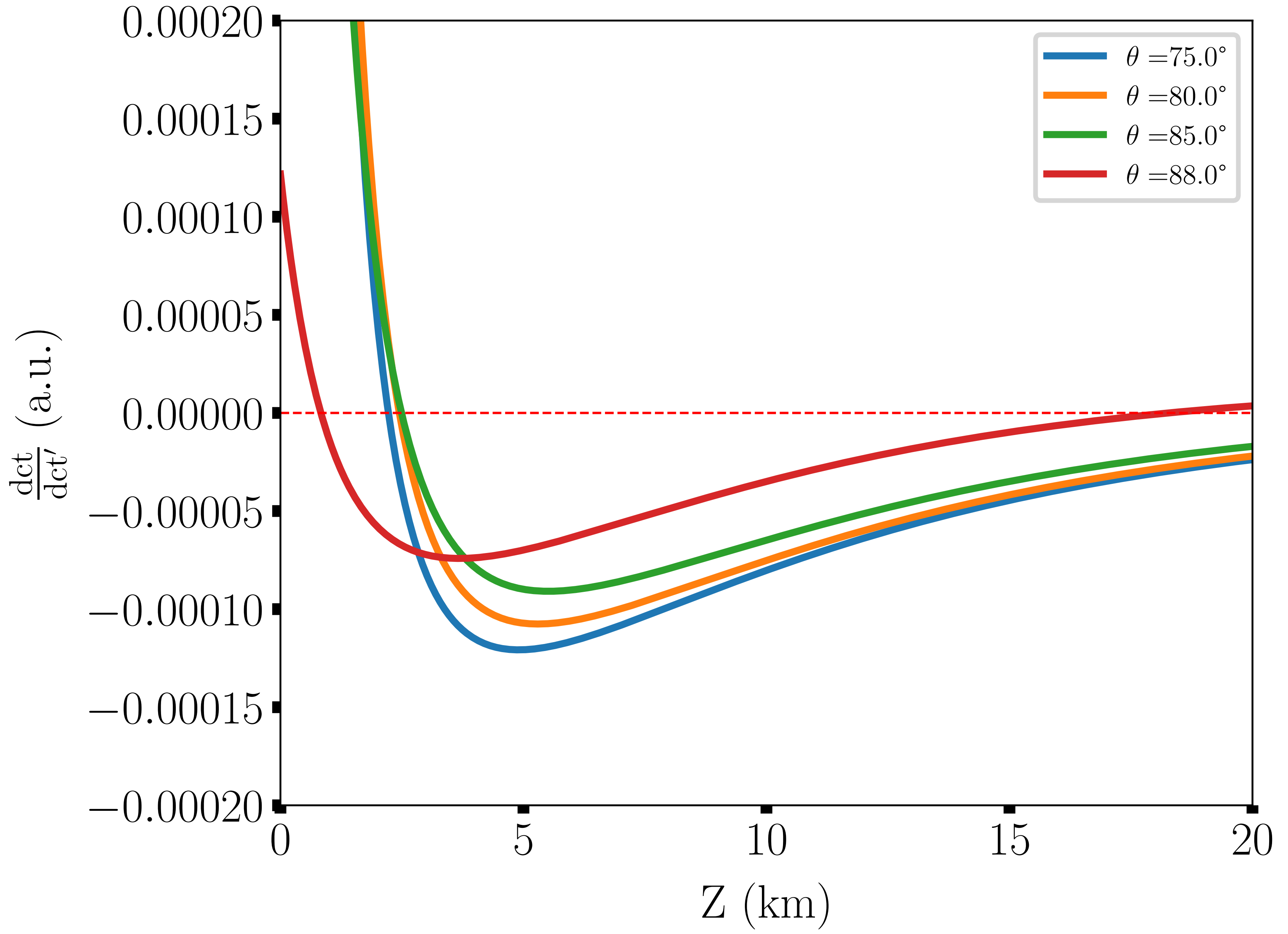
- One emission time → one specific altitude
- For each observer time corresponds at least one emission time

- Inflexion points <-> multi-valued function
- Depends on:
  - shower geometry
  - observer location



→ These inflexion points lead to a “time compression” effects

Time compression effect: boost factor



- Inflexion points == derivatives become infinite
- Derivative → “boost factor” (c.f. vector potential formulae)

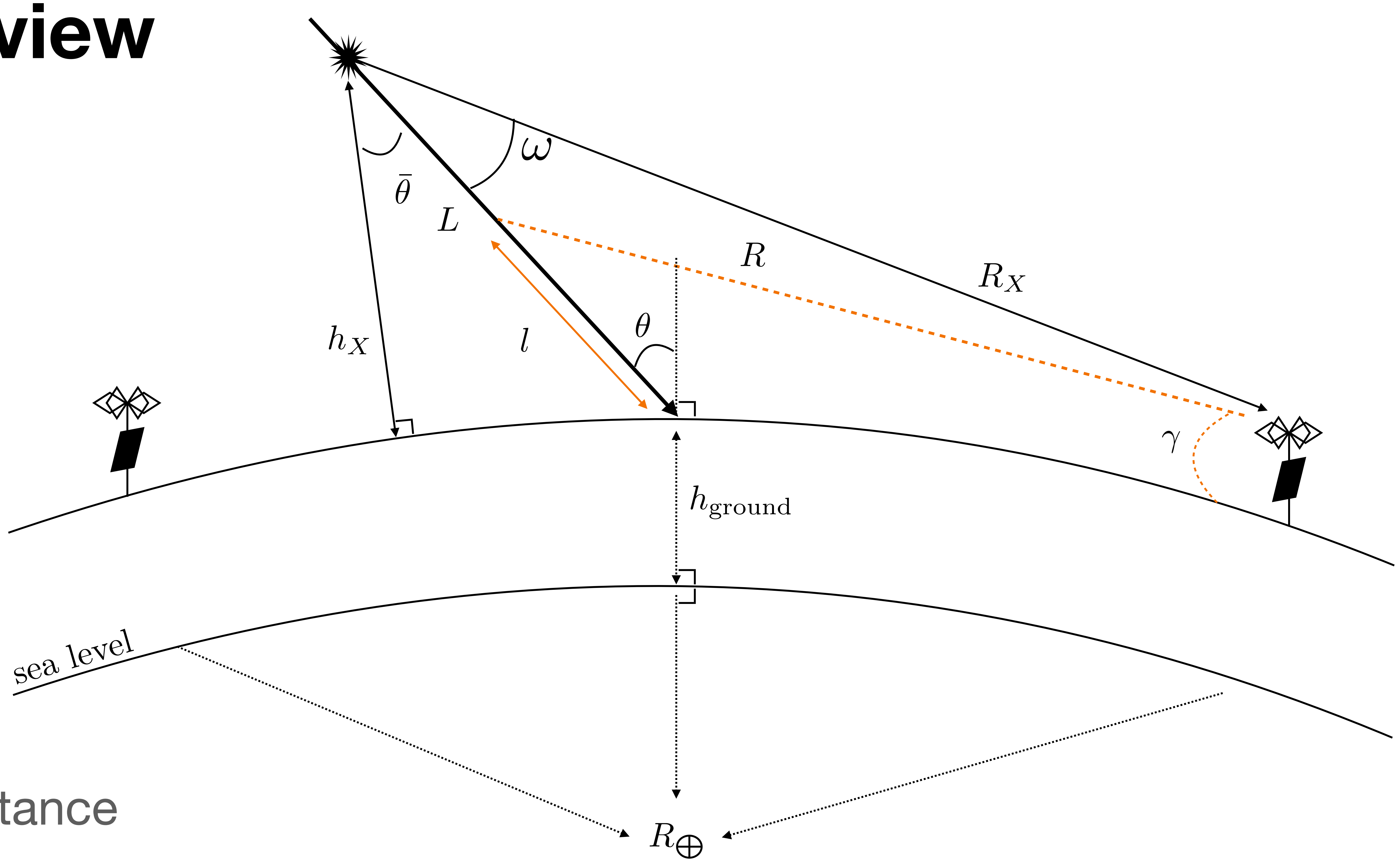
$$\frac{\mathrm{d}ct'}{\mathrm{d}ct} \to \infty$$

$$\Longleftrightarrow \frac{\mathrm{d}ct}{\mathrm{d}ct'} \to 0$$ → numerically more “stable”

→ For each observer location and shower configuration corresponds at least one boosted altitude

→ How to relate this to the Cherenkov effect/angle?

Boost factor: analytical point of view



From the retarded/observer time relation:

$$\frac{\mathrm{d}ct}{\mathrm{d}ct'} = \frac{\mathrm{d}\langle n\rangle}{\mathrm{d}ct'}R + \langle n\rangle\frac{\mathrm{d}R}{\mathrm{d}ct'} + 1$$
$$= \left[\frac{\mathrm{d}\langle n\rangle}{\mathrm{d}R}R + \langle n\rangle\right]\frac{\mathrm{d}R}{\mathrm{d}ct'} + 1$$

→ Variation of: effective refractive index + observer distance

$$\frac{\mathrm{d}R}{\mathrm{d}ct'} = \frac{ct' - R_X\cos(\omega)}{R}$$ "geometry effect"

$$\frac{\mathrm{d}\langle n\rangle}{\mathrm{d}R} = ?$$ "physical effect"

→ We need an analytical expression of the effective refractive index!

Effective refractive index: thermal atmospheric model

- Effective refractive index: $\langle n(R) \rangle = \dfrac{\int_0^R \mathrm{d}r \, n(r)}{\int_0^R \mathrm{d}r}$

  with $n(h) = 1 + k e^{-Ch}$

- Corresponding analytical expression:

$$\langle n(R) \rangle = 1 + \frac{k}{R} \int_0^R e^{-Ch(r)} \, \mathrm{d}r \equiv 1 + \frac{k}{R} I(R)$$

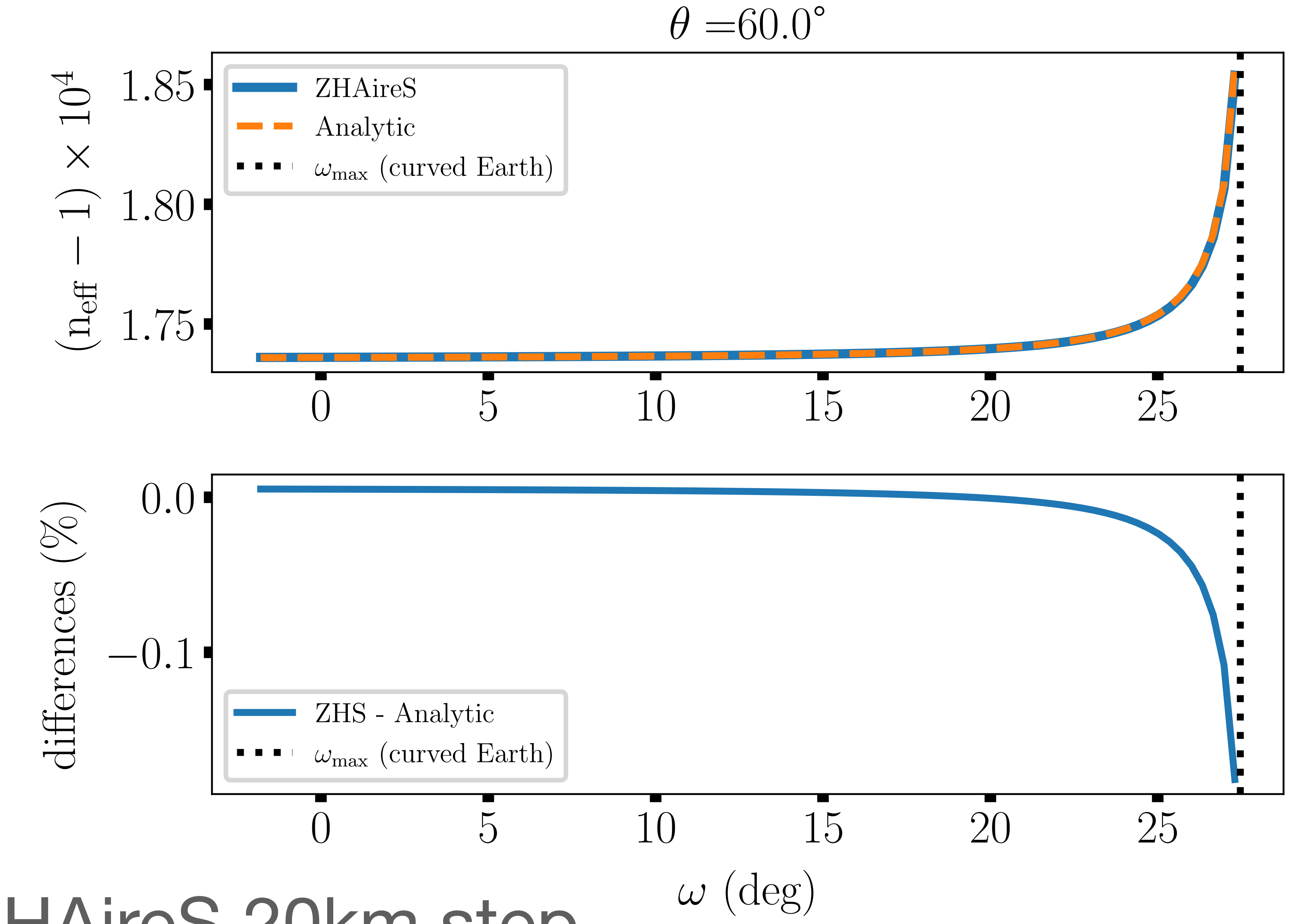
$$I(R) = \sqrt{\frac{\pi R_\oplus}{2C\Gamma}} \, e^{\frac{R_\oplus C \cos^2(\gamma)}{2\Gamma}} \left[\mathrm{erf}\left(\sqrt{\frac{R_\oplus C}{2}\Gamma} \left(\frac{R}{R_\oplus} - \frac{\cos(\gamma)}{\Gamma} \right) \right) + \mathrm{erf}\left(\sqrt{\frac{R_\oplus C \cos^2(\gamma)}{2\Gamma}} \right) \right]$$

→ Depends distance emission observer = R

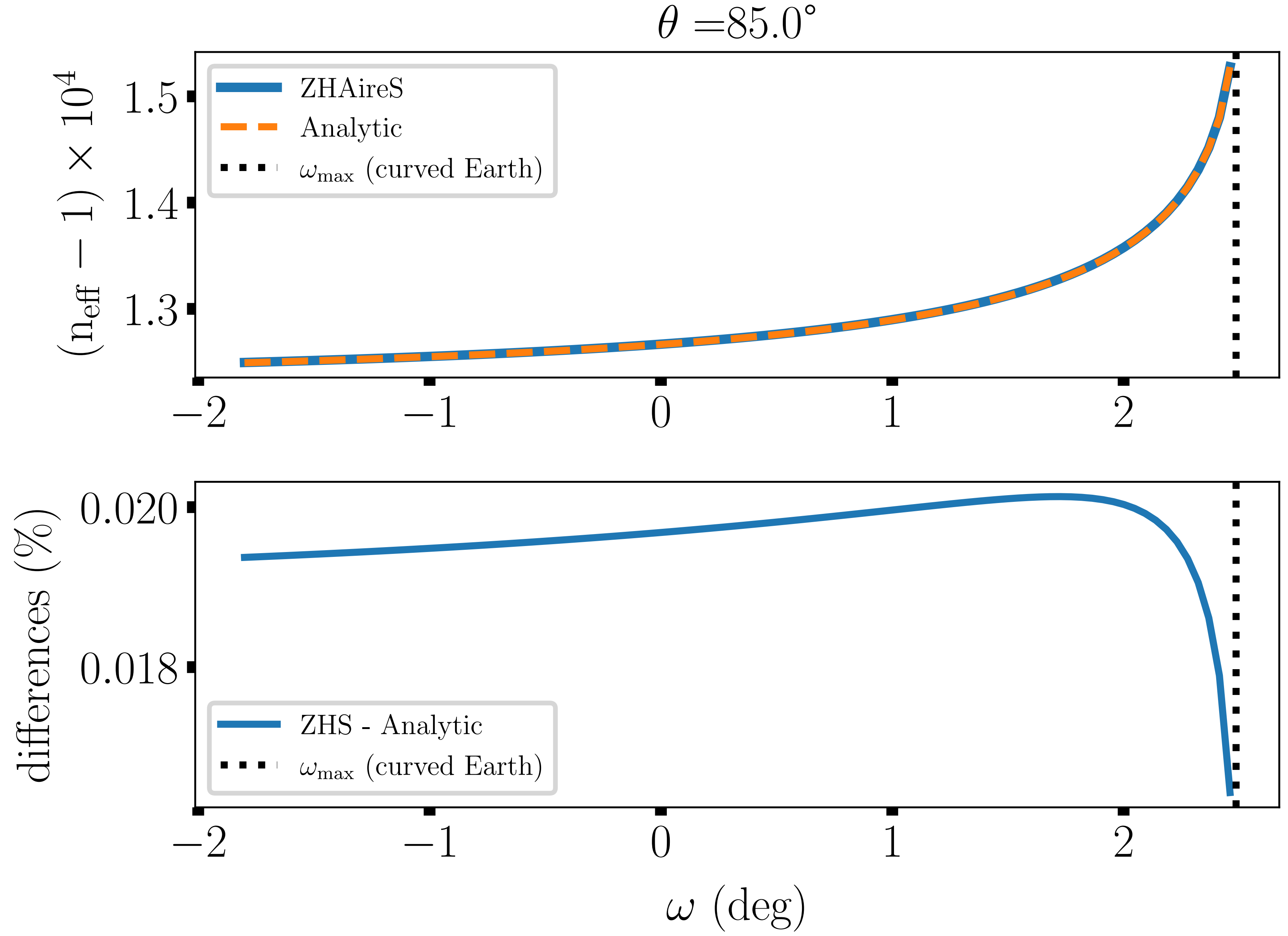
- Derivative:

$$\frac{\mathrm{d}I(R)}{\mathrm{d}R} = I(R) \frac{\cos(\gamma)}{\Gamma} \dot{\gamma} \left(1 + \frac{CR_\oplus}{\Gamma} \right) + e^{CR\left(\cos(\gamma) - \frac{\Gamma R}{2R_\oplus}\right)} \left[1 - \frac{\dot{\gamma}}{\Gamma} \left(\frac{R_\oplus}{\Gamma} + \cos(\gamma) R \right) \right] + \frac{R_\oplus}{\Gamma^2} \dot{\gamma}$$

with $\dot{\gamma} = \dfrac{\mathrm{d}\cos(\gamma)}{\mathrm{d}R} = \dfrac{1}{R_\oplus} - \dfrac{\cos(\gamma)}{R} + \left(\dfrac{h_0}{R_\oplus} + 1 \right) \dfrac{L_X - ct' + R_\oplus \cos(\theta)}{[ct' - R_X \cos(\omega)][h_0 + R_\oplus]}$

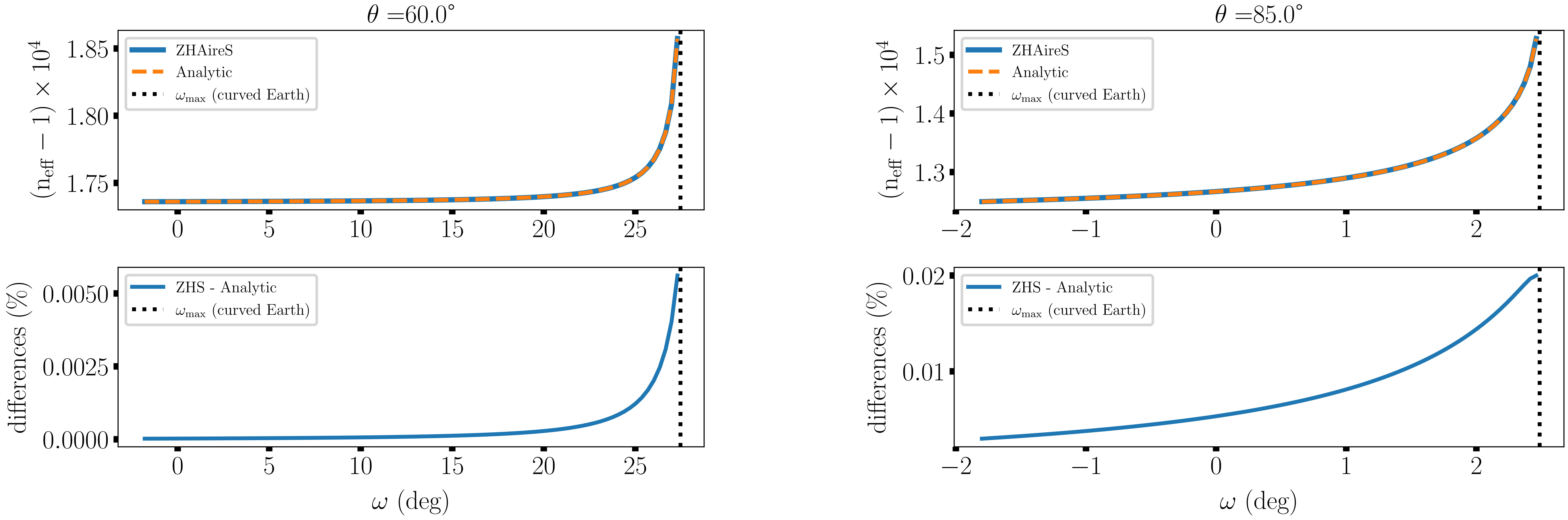


ZHAireS 20km step



Effective refractive index: analytical accuracy

- If we reduce the numerical integration step of ZHAireS down to 1m



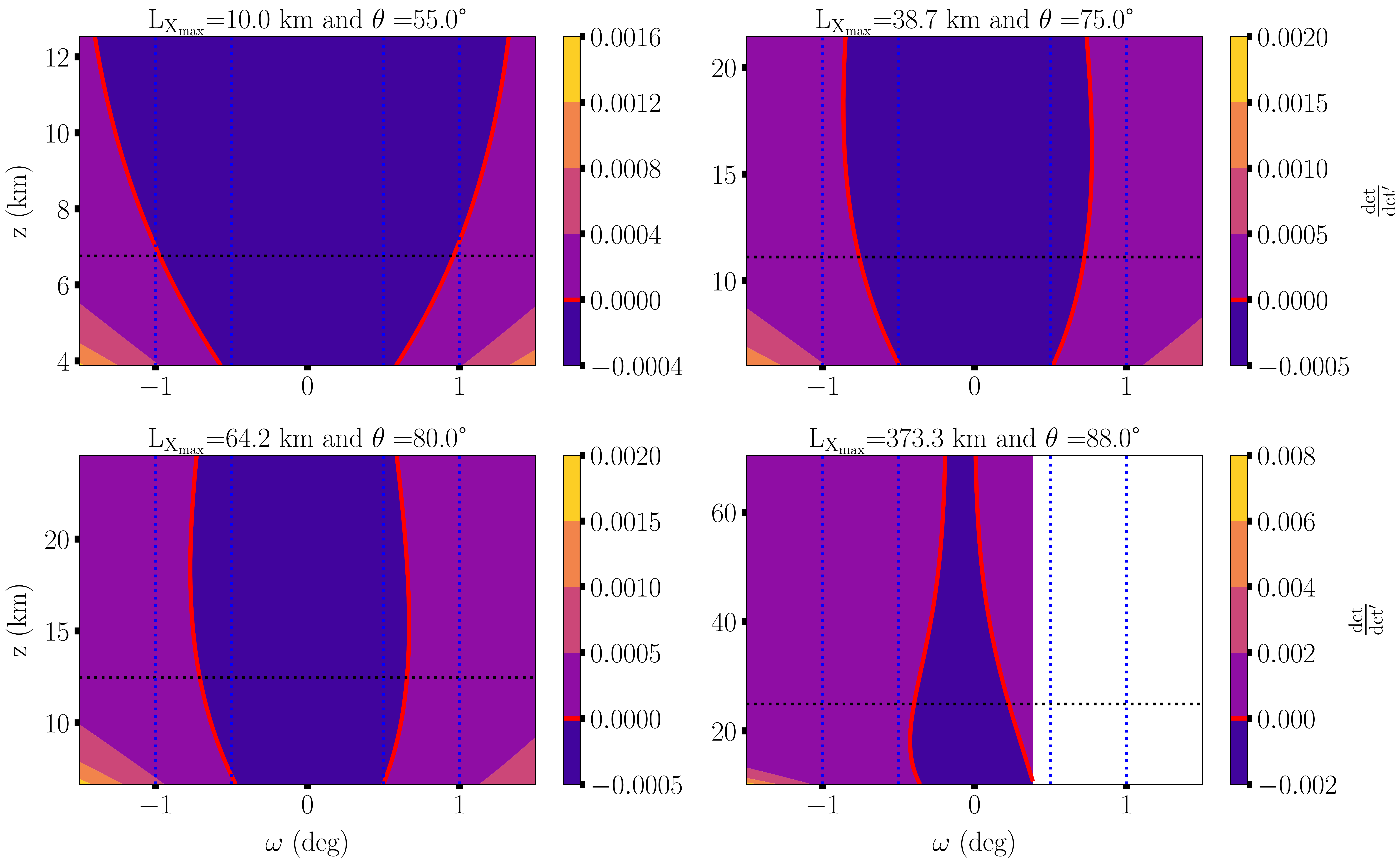
→ Numerical computation converges to the analytical result

Boost factor mapping

- At which altitudes and for which observer the time compression effect is maximal?

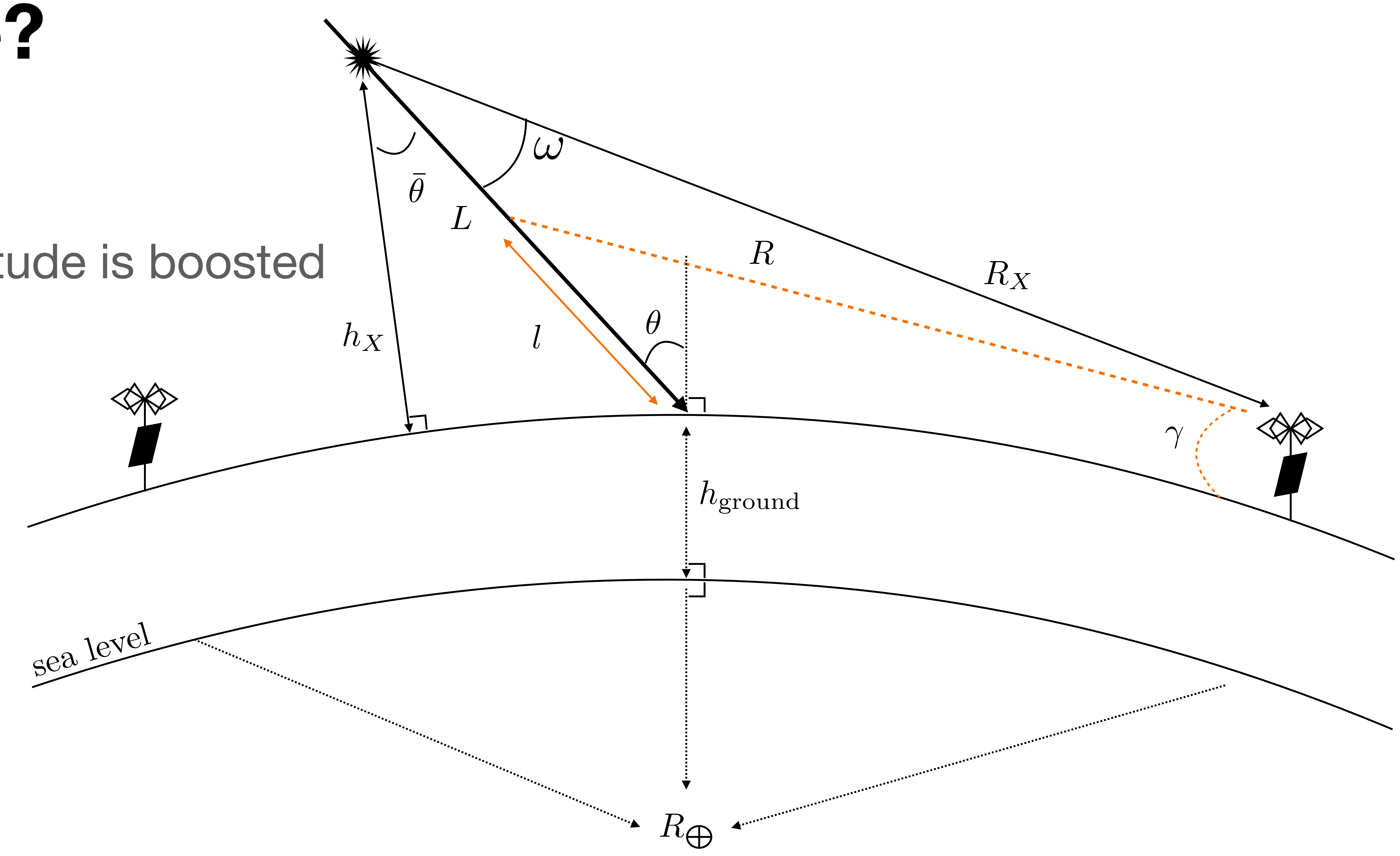
$$\frac{\mathrm{d}ct}{\mathrm{d}ct'} = \frac{\mathrm{d}\langle n\rangle}{\mathrm{d}ct'}R + \langle n\rangle \frac{\mathrm{d}R}{\mathrm{d}ct'} + 1$$
$$= \left[\frac{\mathrm{d}\langle n\rangle}{\mathrm{d}R}R + \langle n\rangle\right]\frac{\mathrm{d}R}{\mathrm{d}ct'} + 1$$

Depends on ct' <-> z and omega



→ Boost factor mapping becomes asymmetrical with shower inclination

What about the Cherenkov angle?



Simple(st) assumption:
→ Cherenkov angle == observer angle where Xmax altitude is boosted

$$\left.\frac{\mathrm{d}ct}{\mathrm{d}ct'}\right|_{\omega=\omega_{\mathrm{Ch}}} = 0$$

Let us consider a constant refractive index: $\langle n \rangle = n$

→ $\frac{\mathrm{d}ct}{\mathrm{d}ct'} = n\frac{ct' - R_X \cos(\omega)}{R} + 1$

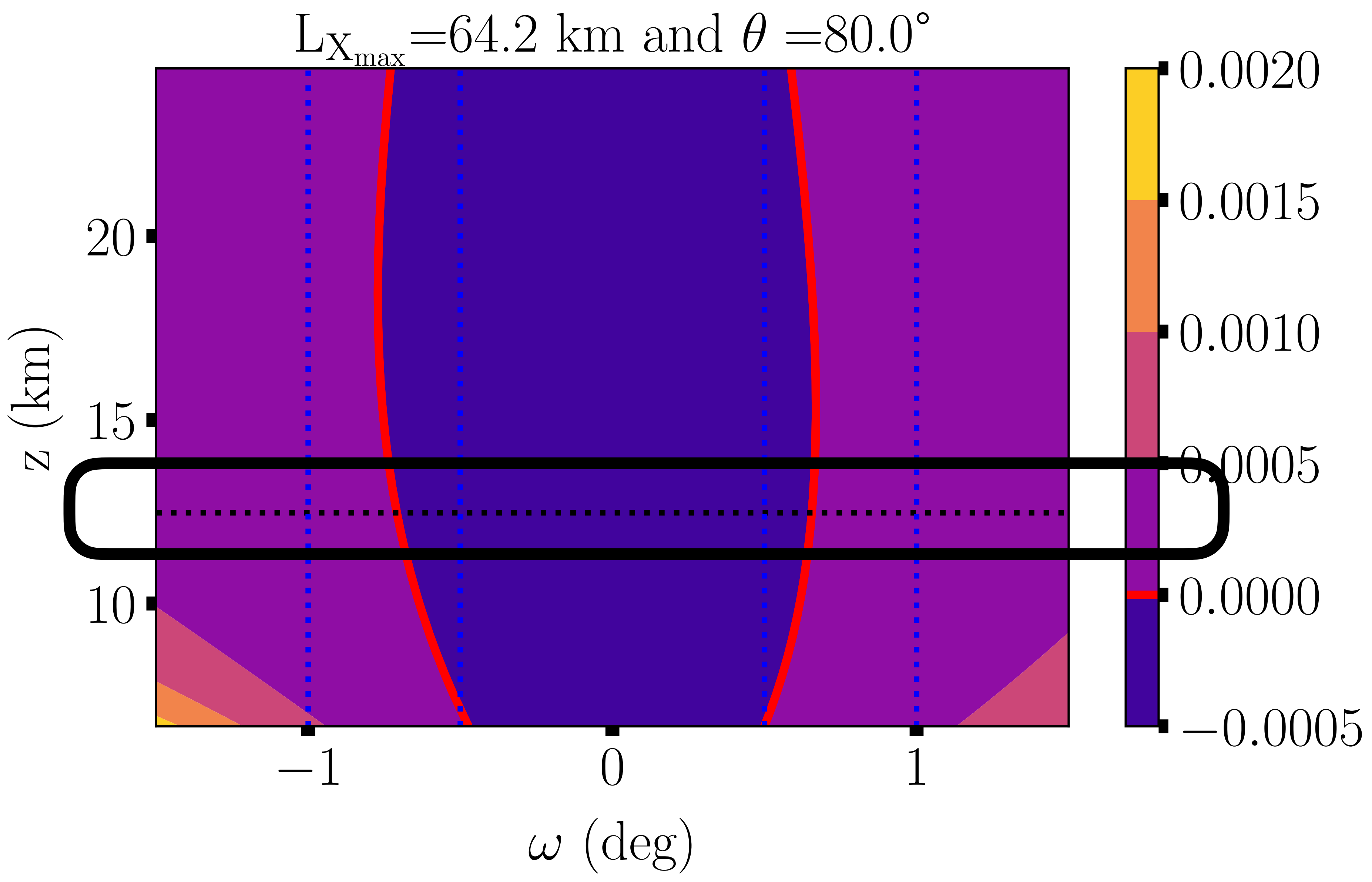
Xmax: $ct' = 0$ -> $-n\cos(\omega) + 1 = 0$

<-> $\omega_{\mathrm{Ch}} = \arccos(1/n)$ → standard result is retrieved when considering a constant index of refraction

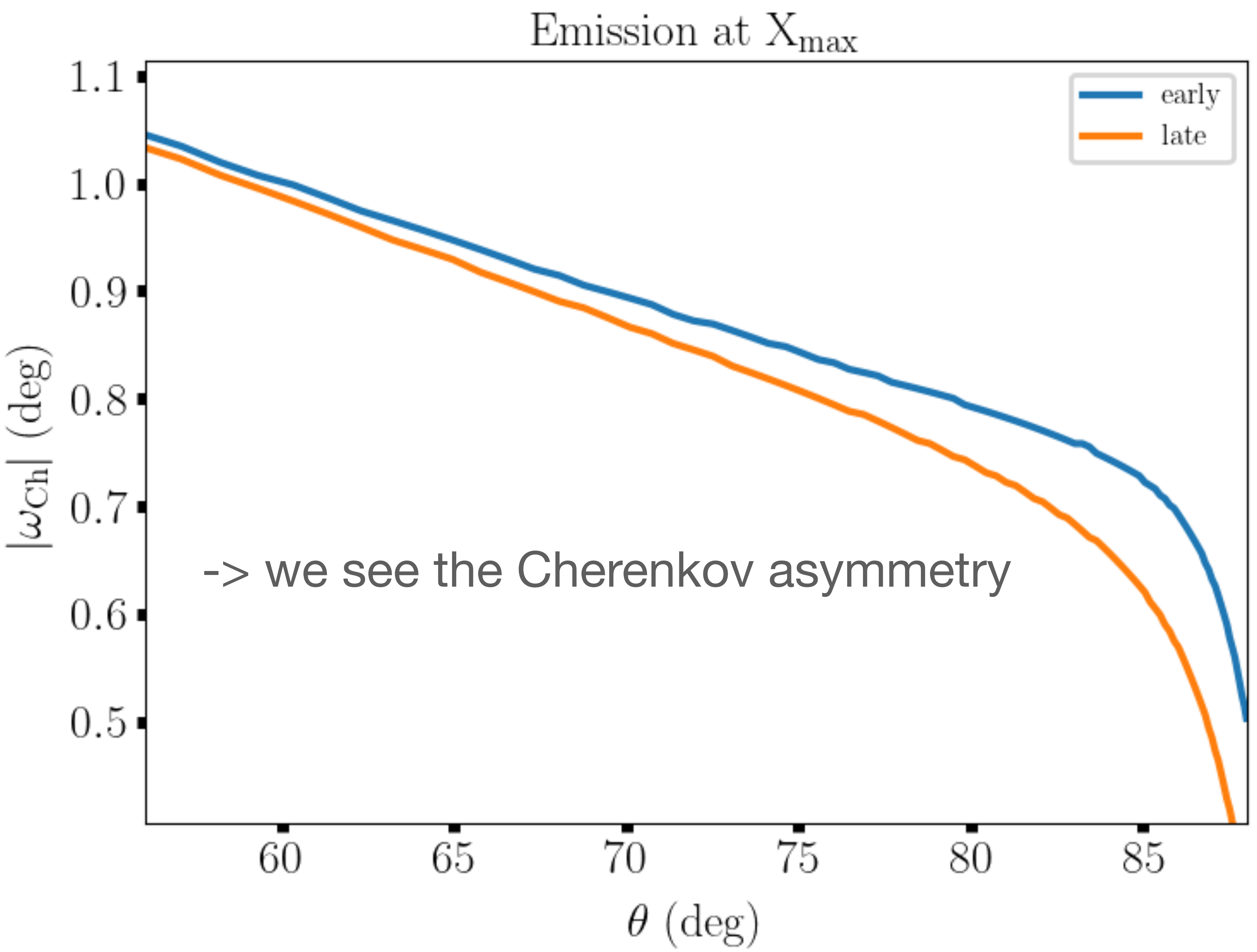
→ What happens with a varying refractive index?

Cherenkov angle(s)

- Select emission at Xmax (ct'=0)
- Find where the inverse boost factor reaches 0 (for omega)



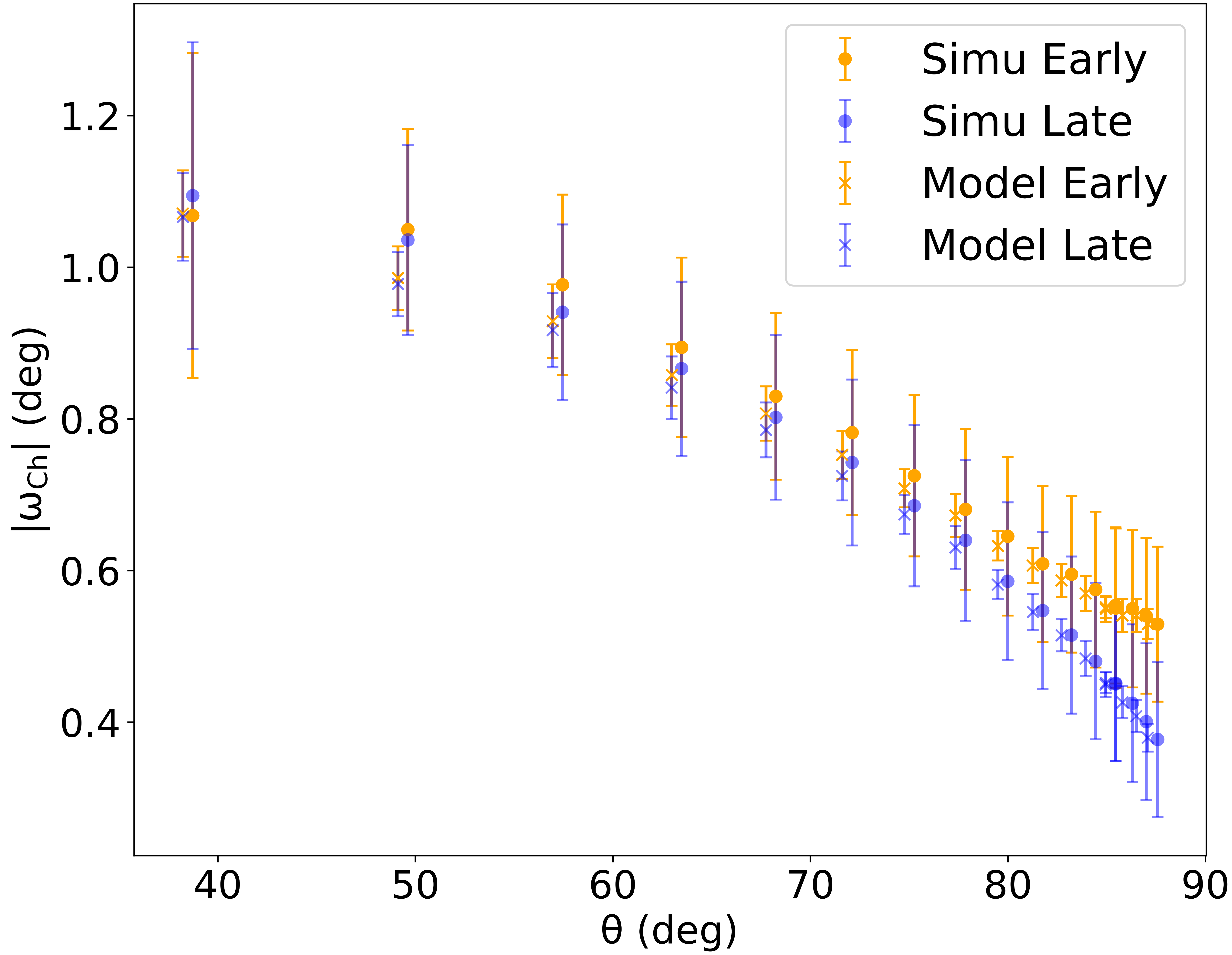
repeated for multiple shower configuration



→ How does it compare to simulations?

Comparison to simulations

- Star shape layout
- ZHAireS
- Error bars: std of distrib “+" antenna step



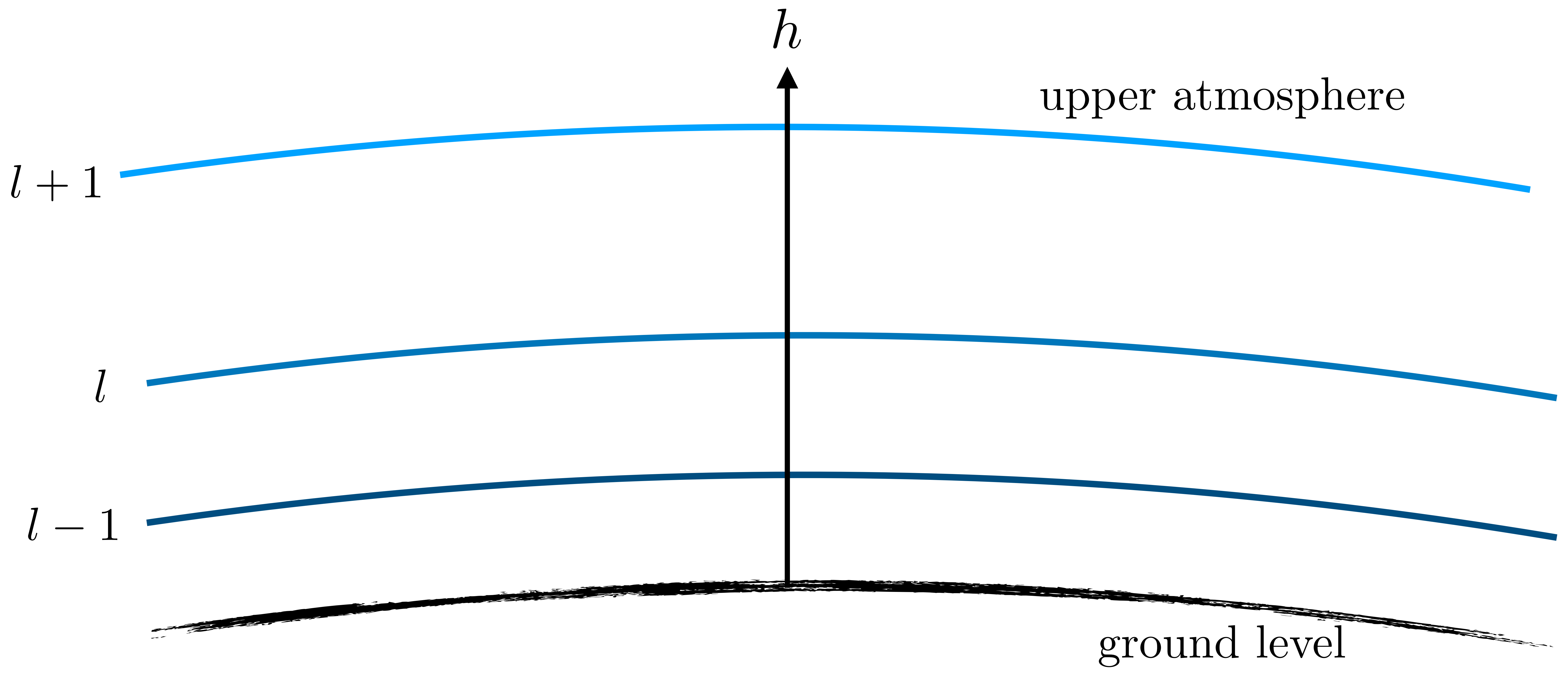
→ Good accuracy between the model and the simulations

Generalisation to layered atmospheric models:

- We can apply the same framework for layered atmospheric models

- Layered refractive index:

$$n_l(h) = 1 + k\frac{b_l}{c_l}e^{-h/c_l}$$



→ These models describe more accurately the variation of refractive index with altitude

Effective refractive index:

- Effective layered refractive index:

$$\langle n(R)\rangle = 1 + \frac{k}{R}\left[\sum_{l=0}^{L}\frac{b_l}{c_l}\int_{R_l}^{R_{l+1}}\mathrm{d}r\, e^{-h(r)/c_l} + \frac{b_L}{c_L}\int_{R_L}^{R}\mathrm{d}r\, e^{-h(r)/c_L}\right]$$

- Analytical formula:

$$\langle n(R)\rangle = 1 + \frac{k}{R}\left[\sum_{l=0}^{L}\mathcal{E}_l(R)\frac{b_l}{c_l}[\mathrm{erf}\,(u_l(R,R_{l+1})) - \mathrm{erf}\,(u_l(R,R_l))] + \mathcal{E}_L(R)\frac{b_L}{c_L}[\mathrm{erf}\,(u_L(R,R)) - \mathrm{erf}\,(u_L(R,R_L))]\right]$$

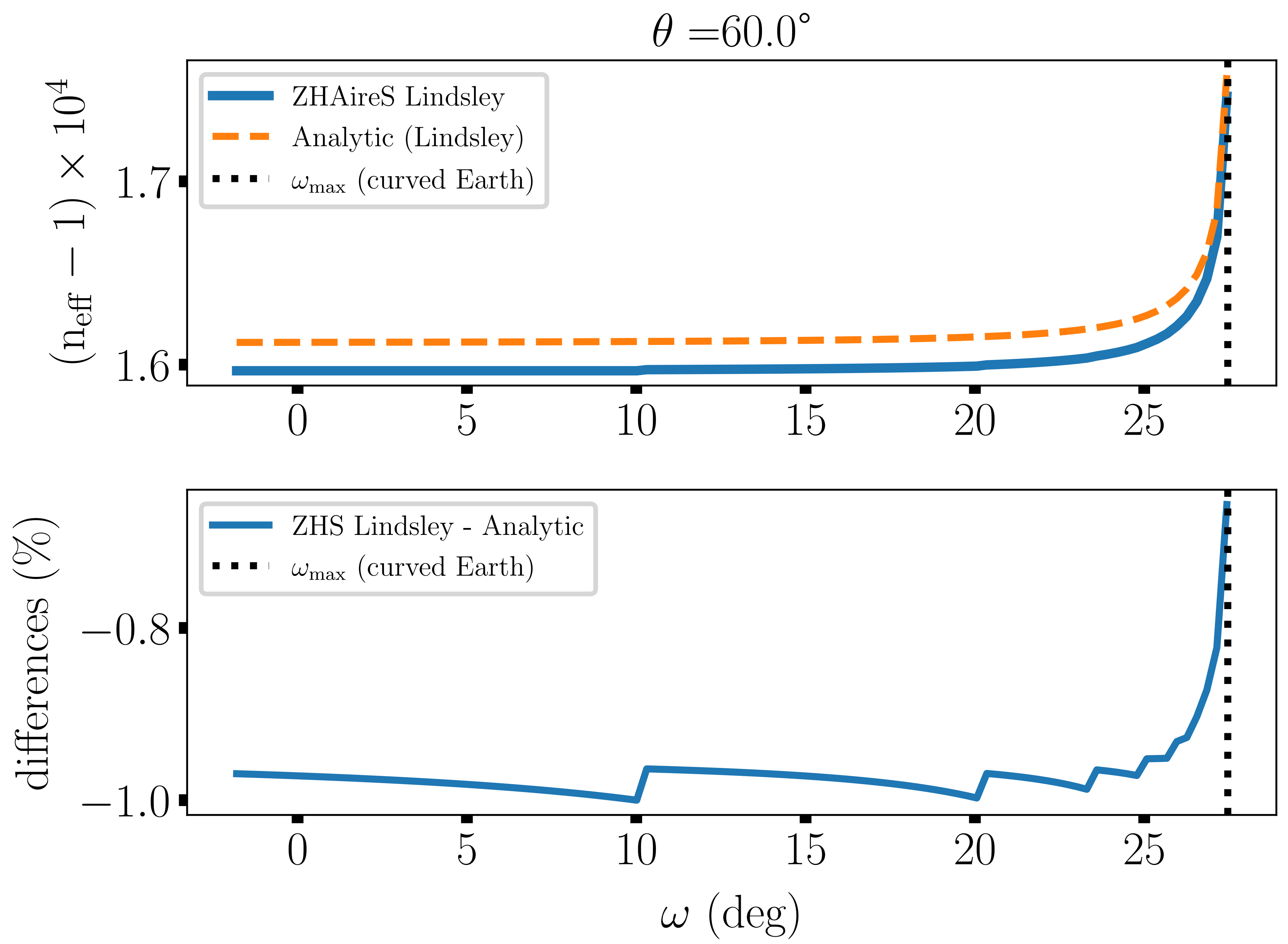
with $\mathcal{E}_l(R) = \sqrt{\frac{\pi R_\oplus c_l}{2\Gamma}}\, e^{\frac{R_\oplus \cos^2(\gamma)}{2c_l\Gamma}}$ and $u_l(R,x) = \sqrt{\frac{R_\oplus \Gamma}{2c_l}}\left(\frac{x}{R_\oplus} - \frac{\cos(\gamma)}{\Gamma}\right)$

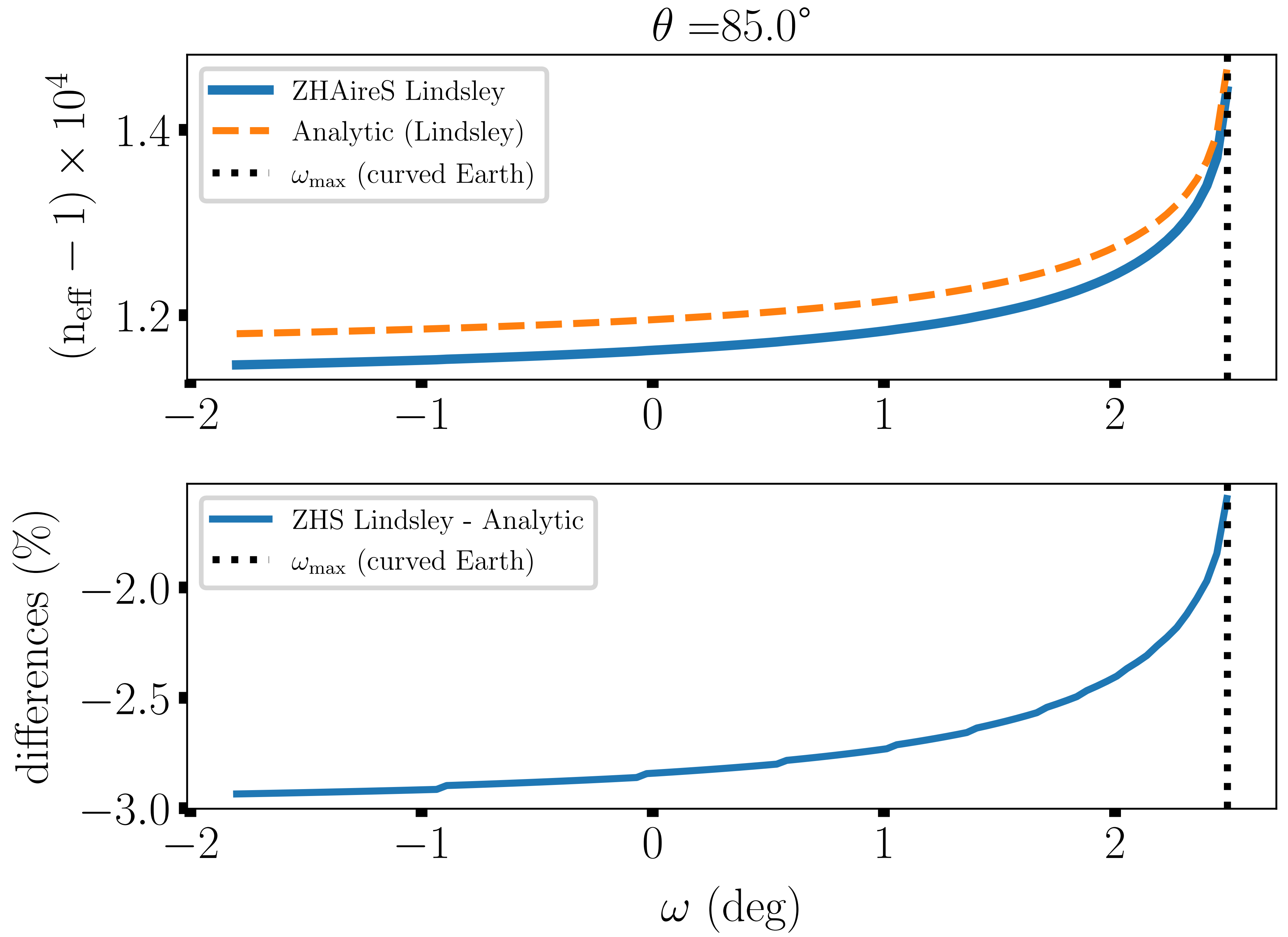
- Derivative:

$$\frac{\mathrm{d}I(R)}{\mathrm{d}R} = \sum_{l=0}^{L}\frac{b_l}{c_l}\left[\frac{\mathrm{d}\mathcal{E}_l(R)}{\mathrm{d}R}[\mathrm{erf}\,(u_l(R,R_{l+1})) - \mathrm{erf}\,(u_l(R,R_l))] + \frac{2}{\sqrt{\pi}}\mathcal{E}_l(R)\left[\frac{\mathrm{d}u_l(R,R_{l+1})}{\mathrm{d}R}e^{-u_l(R,R_{l+1})^2} - \frac{\mathrm{d}u_l(R,R_l)}{\mathrm{d}R}e^{-u_l(R,R_l)^2}\right]\right]$$
$$+ \frac{b_L}{c_L}\left[\frac{\mathrm{d}\mathcal{E}_L(R)}{\mathrm{d}R}[\mathrm{erf}\,(u_L(R,R)) - \mathrm{erf}\,(u_L(R,R_L))] + \frac{2}{\sqrt{\pi}}\mathcal{E}_L(R)\left[\frac{\mathrm{d}u_L(R,R)}{\mathrm{d}R}e^{-u_L(R,R)^2} - \frac{\mathrm{d}u_L(R,R_L)}{\mathrm{d}R}e^{-u_L(R,R_L)^2}\right]\right]$$

with $\frac{\mathrm{d}\mathcal{E}_l(R)}{\mathrm{d}R} = \mathcal{E}_l(R)\frac{\cos(\gamma)}{\Gamma}\dot{\gamma}\left(1 + \frac{R_\oplus}{c_l\Gamma}\right)$ $\frac{\mathrm{d}u_l(R,x)}{\mathrm{d}R} = \sqrt{\frac{R_\oplus}{2c_l\Gamma}}\left[\frac{\mathrm{d}x}{\mathrm{d}R}\frac{\Gamma}{R_\oplus} - \frac{\dot{\gamma}}{\Gamma}\left(1 + \cos(\gamma)\Gamma\frac{x}{R_\oplus}\right)\right]$

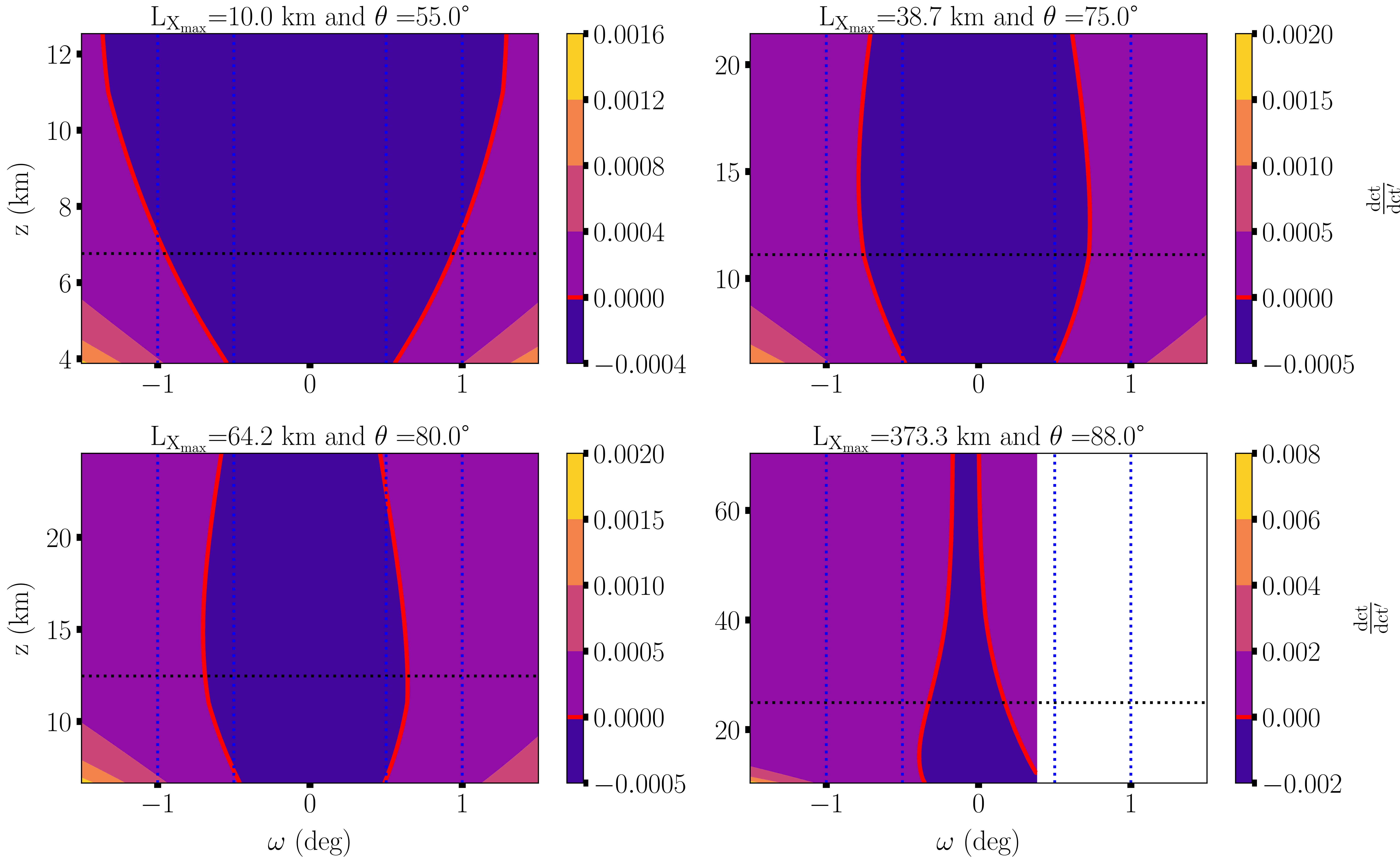
and $\frac{\mathrm{d}R_l}{\mathrm{d}R} = R_\oplus\dot{\gamma}\left(1 + \frac{R_\oplus\cos(\gamma)}{\sqrt{R_\oplus^2\cos^2(\gamma) + h_l^2 + 2h_l R_\oplus}}\right)$





Generalisation to layered atmospheric models:

- Here: Lindsley model



→ Slight changes compared to the thermal atmospheric model

Comparison to simulations

- Star shape layout
- ZHAireS
- Error bars: std of distrib “+" antenna step

Well… coming soon

Conclusion and perspectives

The Cherenkov effects can be accurately described using propagation effect principles
- This analytical model allows to:
  - compute the Cherenkov angles
  - to reproduce the asymmetry seen in simulations
- The analytical expressions for the effective refractive index are obtained:
  - with relative errors < 3%
  - and numerical computation times improved by a factor ~2 to ~O(10,000) (on Python!)

Possible future works:
- Implement the Cherenkov angle computation in reconstruction methods (e.g., ADF)
- Develop new reconstruction methods based on this model?
- Investigate the implementation of the n_eff computation into simulations