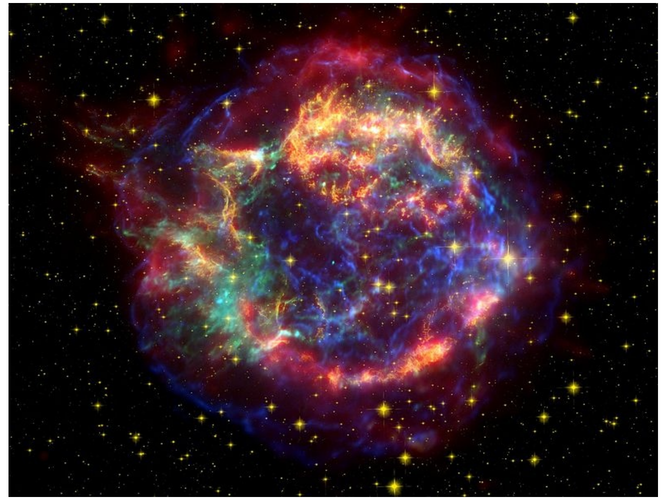


Part II: Gravitational Waves from Core-Collapse Supernovae

Australian Government
Australian Research Council

Bernhard Müller
Monash University

First IGWN Symposium on Core Collapse Supernova Gravitational Wave Theory and Detection

Gravitational Waves from Core-Collapse Supernovae

mass quadrupole moment (transverse-trace free component)

Asymmetry parameter

radius

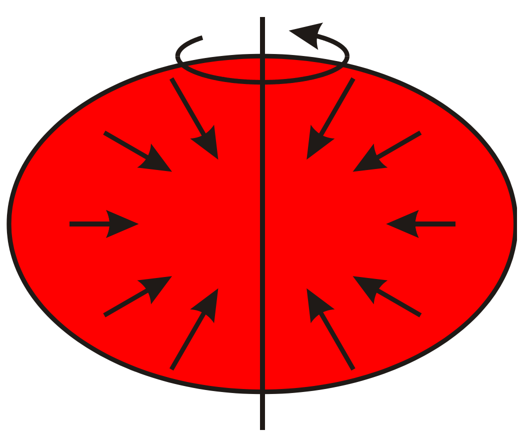
typical frequency

$$h \sim \frac{2G}{c^4 r}\frac{d^2 I}{dt^2} \sim \frac{2G}{c^4 r}\epsilon M R^2 f^2$$

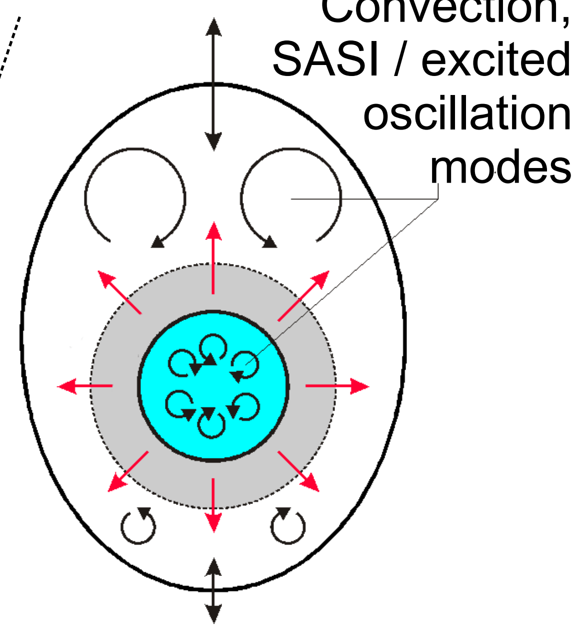
dimensionless strain

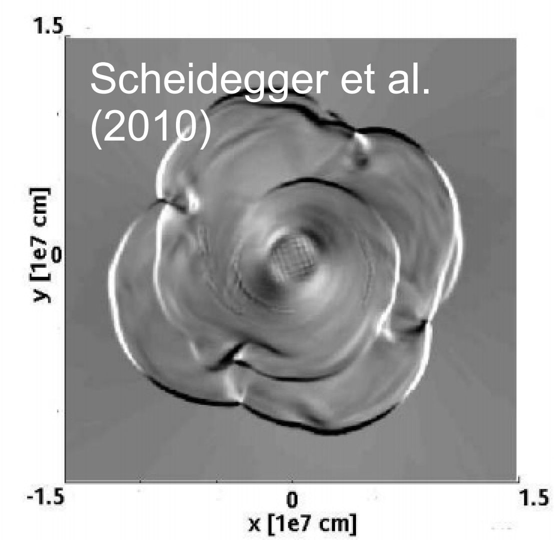
distance

mass involved



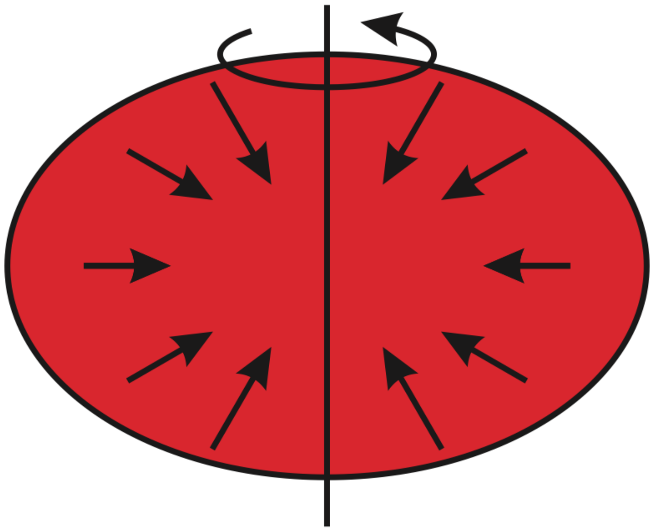
Rotational collapse
Bounce signal

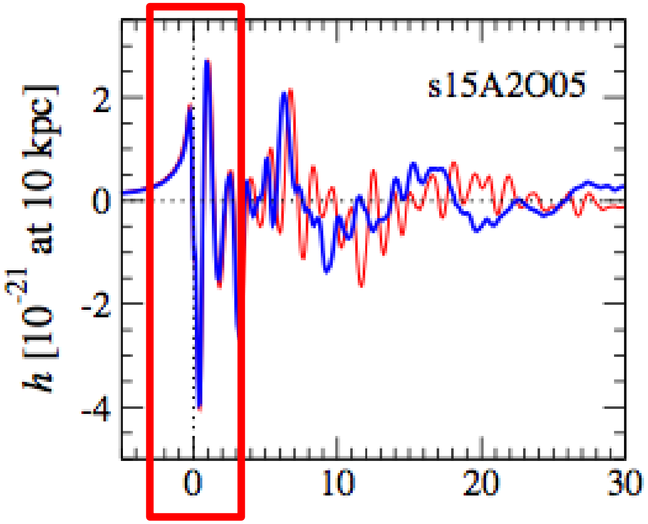


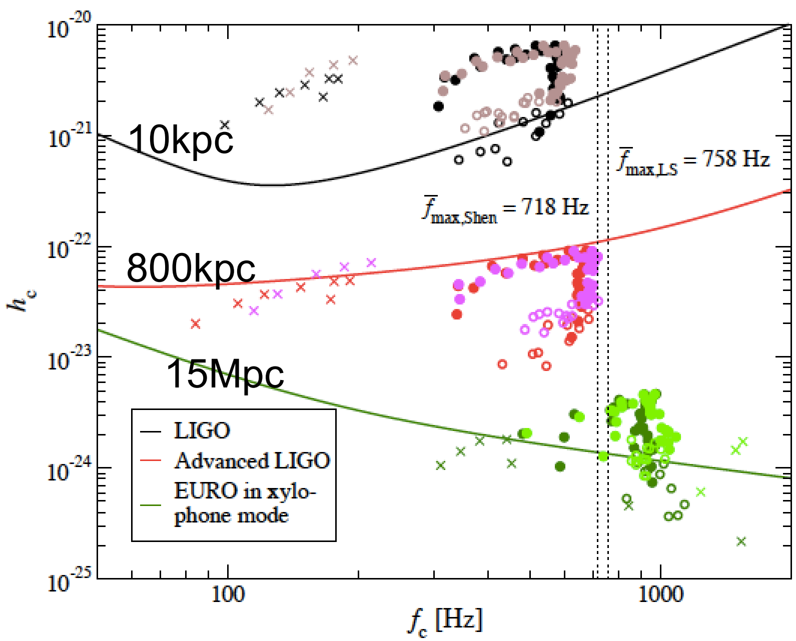


Rotationally-driven instabilities & modes (low T/W, r-mode)

Bounce Signal





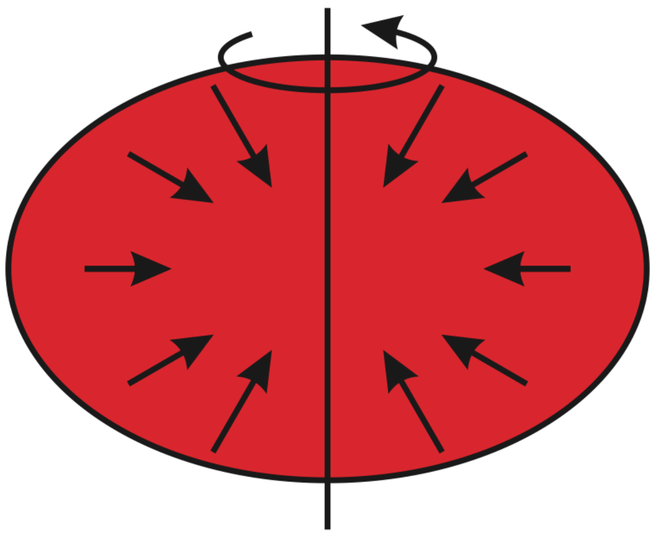


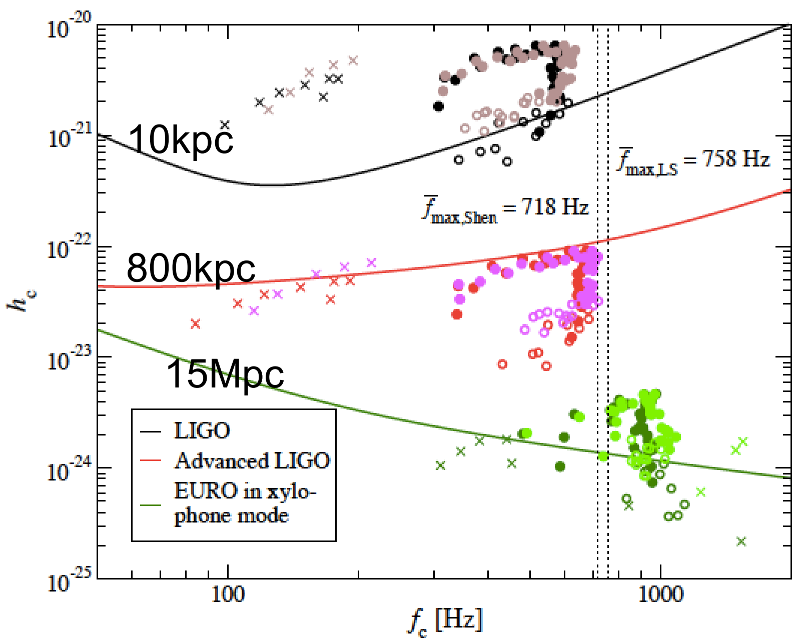
Characteristic strain & frequency for different progenitors, EoS & rotation rates (Dimmelmeier et al. 2008)

Bounce signal is essentially *f*-mode (Fuller et al. '14): very regular shape, amenable to template-based searches: $f \sim 1/2\pi \sqrt{(G\rho_c)}$

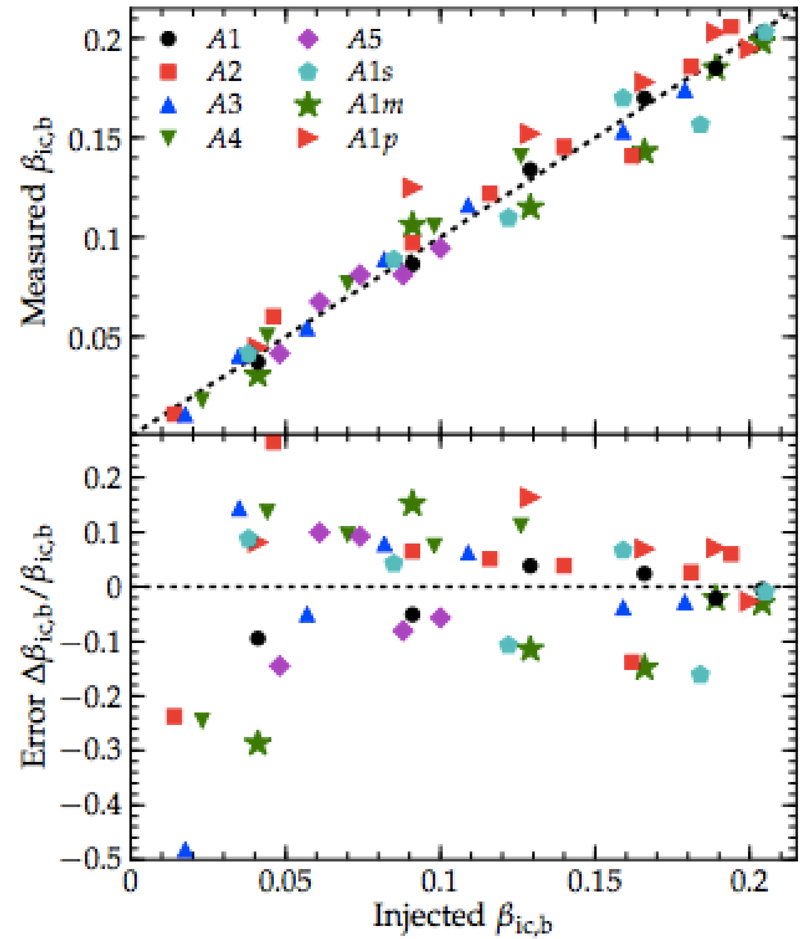
Detectability limit: of order ~40kpc for Advanced LIGO for initial core rotation periods of ~seconds (see, e.g., Logue et al. 2012, Hayama et al. 2015, Gossan et al. 2016)

Bounce Signal





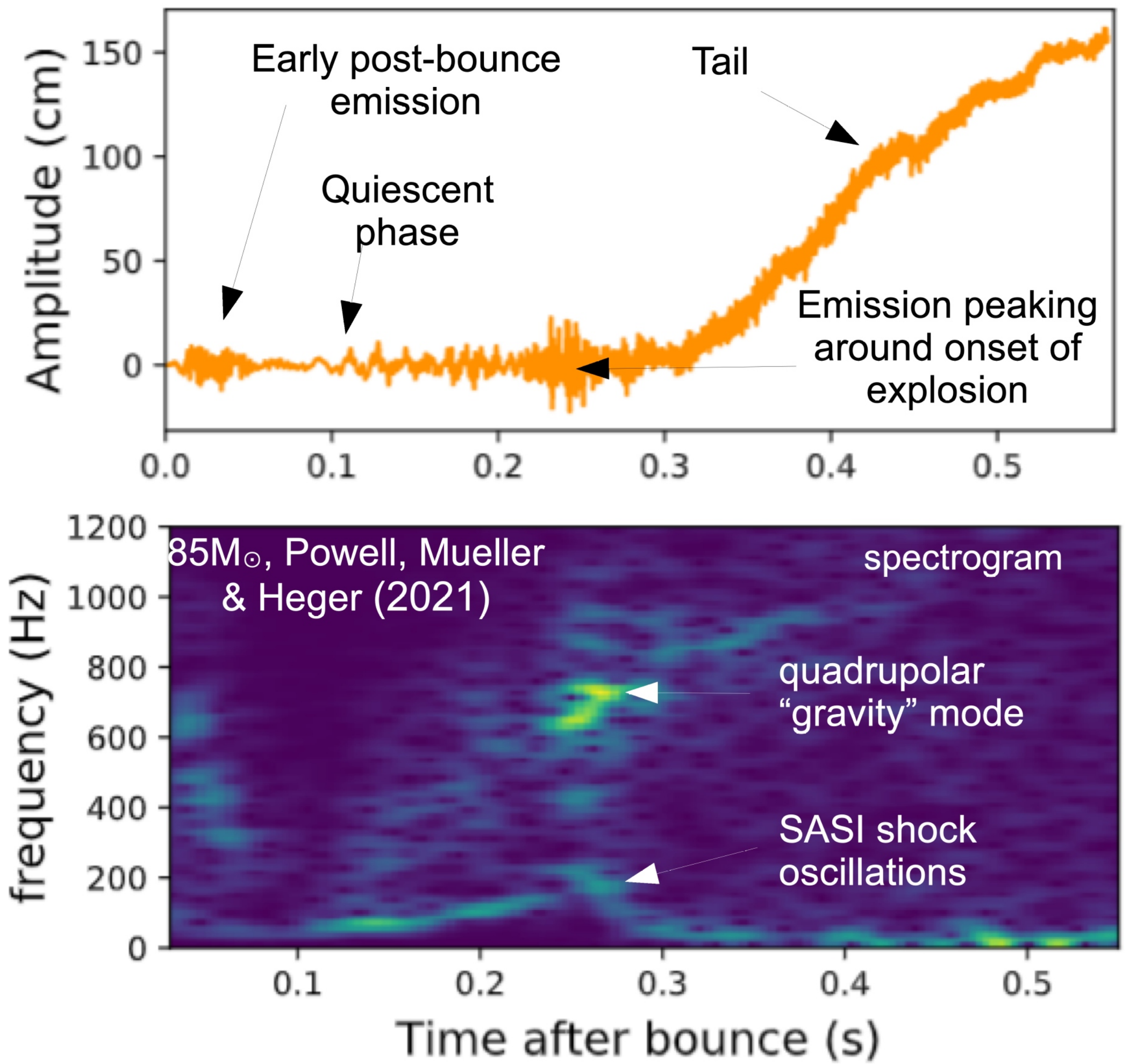
Characteristic strain & frequency for different progenitors, EoS & rotation rates (Dimmelmeier et al. 2008)



Abdikamlov et al. (2014): Inferred β=T/W in progenitor from prospective signal

At ~10kpc, the initial period can be constrained to within ~20%

“Standard” Post-Bounce Gravitational Wave Emission



Waveform & spectrogram features firmly established by many groups (Oak Ridge, Princeton, Fukuoka/NAOJ/AEI/Kyoto, Monash/Garching, Stockholm/MSU, ...)

Tools: Linear Perturbation Theory

- Time-frequency analysis started to reveal distinct noise rather than broad-band noise ~10 years ago
- Rigorous approach to mode structure: linear perturbation theory (Torres-Forne et al. 2018, Morozova et al. 2018...):

$$\partial_r \eta_r + \left[\frac{2}{r} + \frac{1}{\Gamma_1}\frac{\partial_r P}{P} + 6\frac{\partial_r \psi}{\psi}\right]\eta_r + \frac{\psi^4}{\alpha^2 c_s^2}\left(\sigma^2 - \mathcal{L}^2\right)\eta_\perp = 0, \tag{31}$$

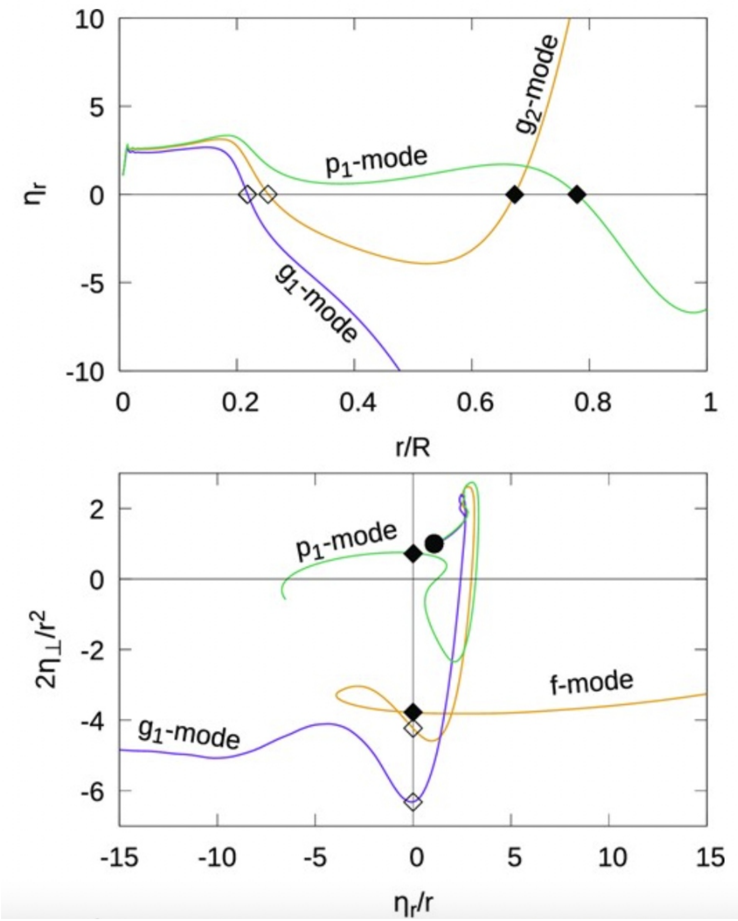
$$\partial_r \eta_\perp - \left(1 - \frac{\mathcal{N}^2}{\sigma^2}\right)\eta_r + \left[\partial_r \ln q - G\left(1 + \frac{1}{c_s^2}\right)\right]\eta_\perp = 0, \tag{32}$$

Torres-Forne et al. (2018)

- Asymptotic theory (Mueller et al. 2013) sufficient for big picture

Linear Perturbation Theory: Limitations

- Correct outer boundary condition is not trivial
- Perturbations outside PNS are **not** adiabatic
- Rapid rotation or strong magnetic fields would require modifications
- Linear theory does not address mode excitation
- Minor: Different mode classifications exist → avoid misunderstandings



Rodriguez et al. (2023): Comparison of mode classification schemes

Phases: Early Post-Bounce Phase

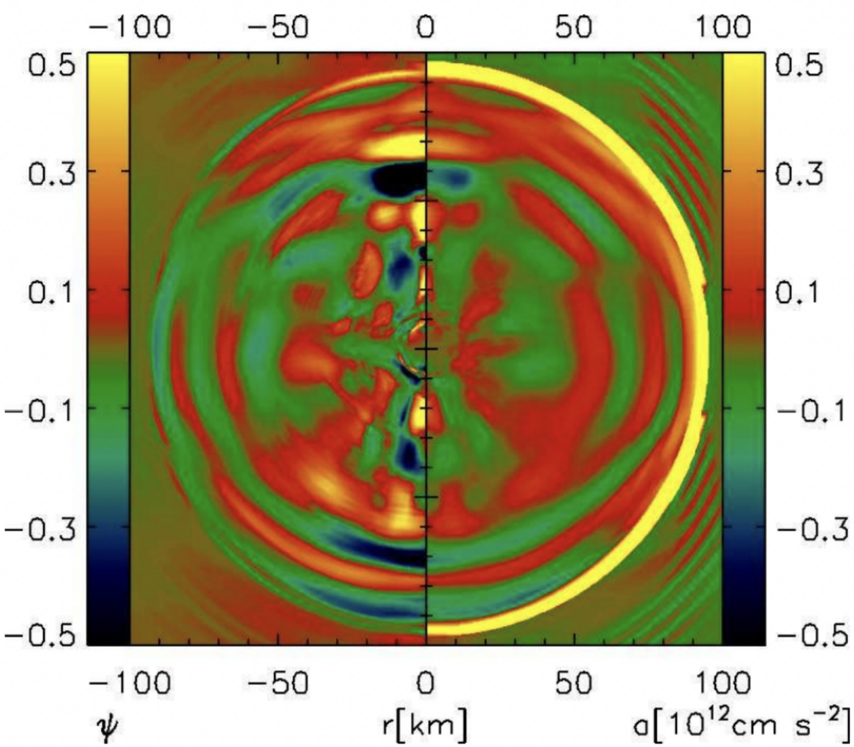
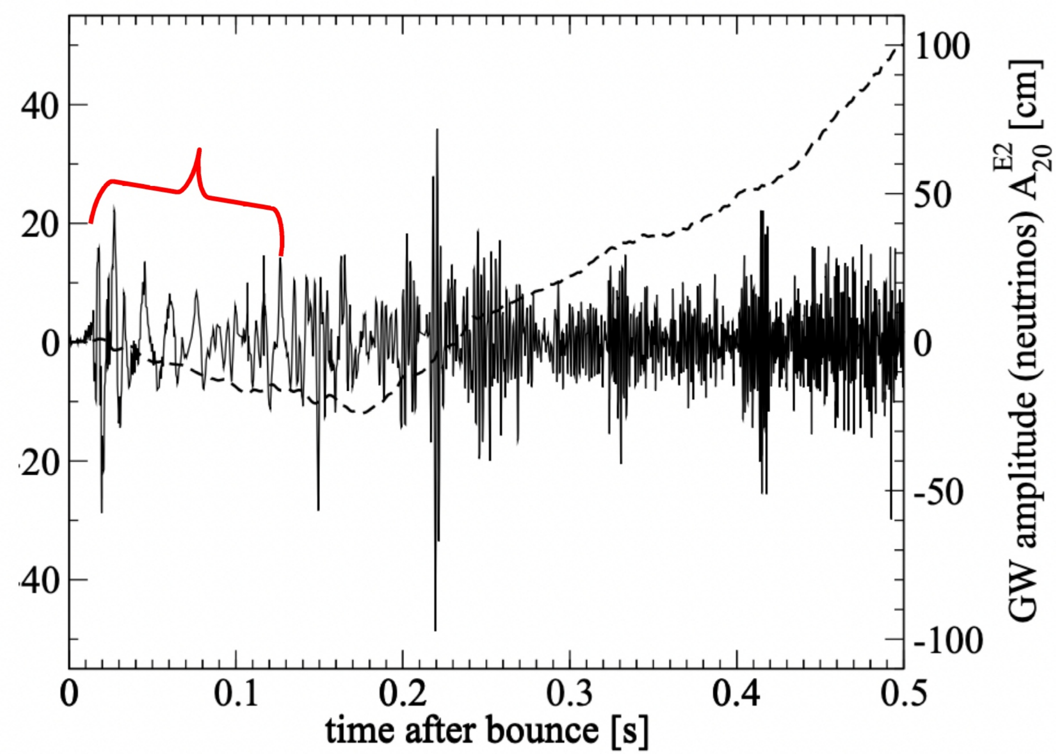


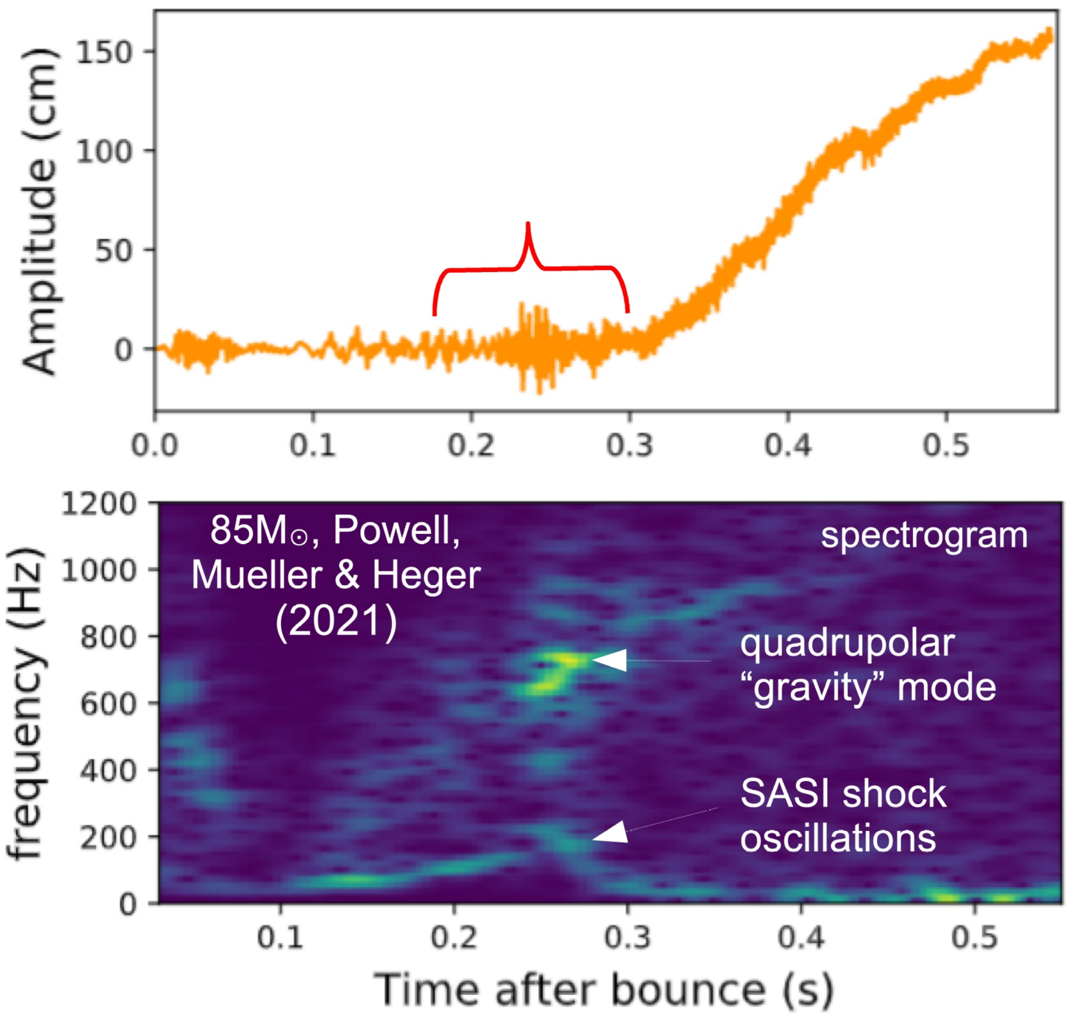
Figure 7. The dimensionless integrand ψ in the quadrupole formula (1) for the matter signal (left half of figure) and the time derivative $a = \partial v_r / \partial t$ (right half) of the radial velocity field 22 ms after bounce for model G15.

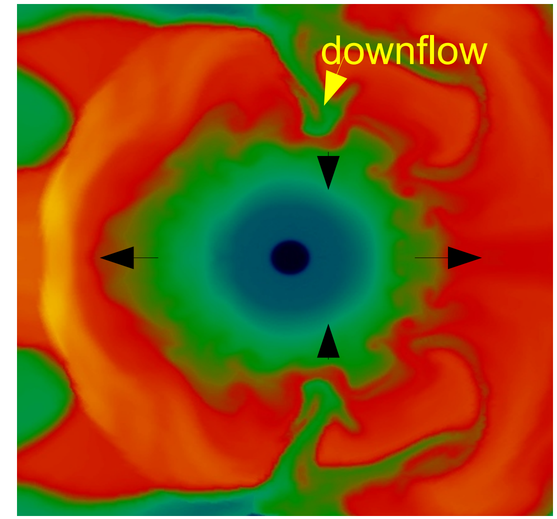


Mueller et al. (2013)

- Prompt convection quickly mixes unstable region and subsides
- Kicks off “ringing” in the shock (“early SASI”)
- GW emission due to acoustic weaves between PNS and shock
- **Caution:** Dynamics of prompt convection depends on transport treatment (breakout burst), seed asymmetries, numerical dissipation...

Dominant High-Frequency Signal





neutron mass

electron antineutrino mean energy ~ neutron star surface temperature

neutron star mass

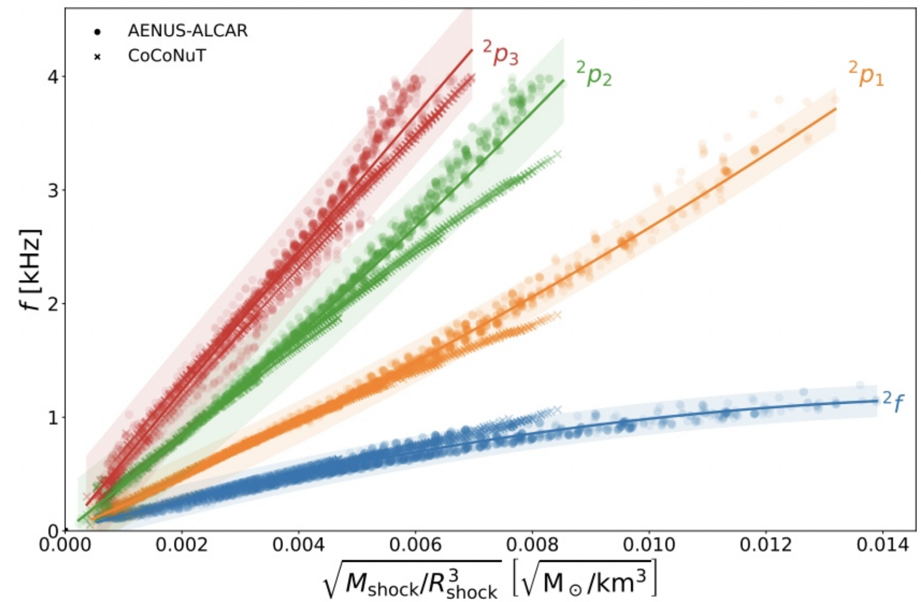
$$f_g \approx \frac{1}{2\pi}\frac{GM}{R^2}\sqrt{1.1\frac{m_n}{\langle E_\nu \rangle}}\left(1-\frac{GM}{Rc^2}\right)^2$$

neutron star radius

- Turbulent motions inside and outside the PNS can excite the oscillations of the stable surface region
- Frequency roughly set by local Brunt-Väisälä frequency
- Dominant mode usually classified as f-mode or low-order g-mode
- GR treatment & monopole vs. multi-D gravity matter for frequency

Relation to PNS Parameters

- Other weak modes typically present ("p-mode forest" above dominant mode)
- Various simple scaling laws to PNS mass and radius proposed for mode frequencies (Torres-Forne et al. '19, Sotani et al. '21)
- Potential for inference of PNS parameters even if temperature is not known from neutrinos
- How "universal" are these – and can we account for confounders (rotation, etc.)?



mode	x
^2f	$\sqrt{M_{\rm shock}/R^3_{\rm shock}}$
$^2p_1$	$\sqrt{M_{\rm shock}/R^3_{\rm shock}}$
$^2p_2$	$\sqrt{M_{\rm shock}/R^3_{\rm shock}}$
$^2p_3$	$\sqrt{M_{\rm shock}/R^3_{\rm shock}}$
$^2g_1$	$M_{\rm pns}/R^2_{\rm pns}$
$^2g_2$	$M_{\rm pns}/R^2_{\rm pns}$
$^2g_3$	$\sqrt{M_{\rm shock}/R^3_{\rm shock}}\ p_C/\rho_C^{2.5}$

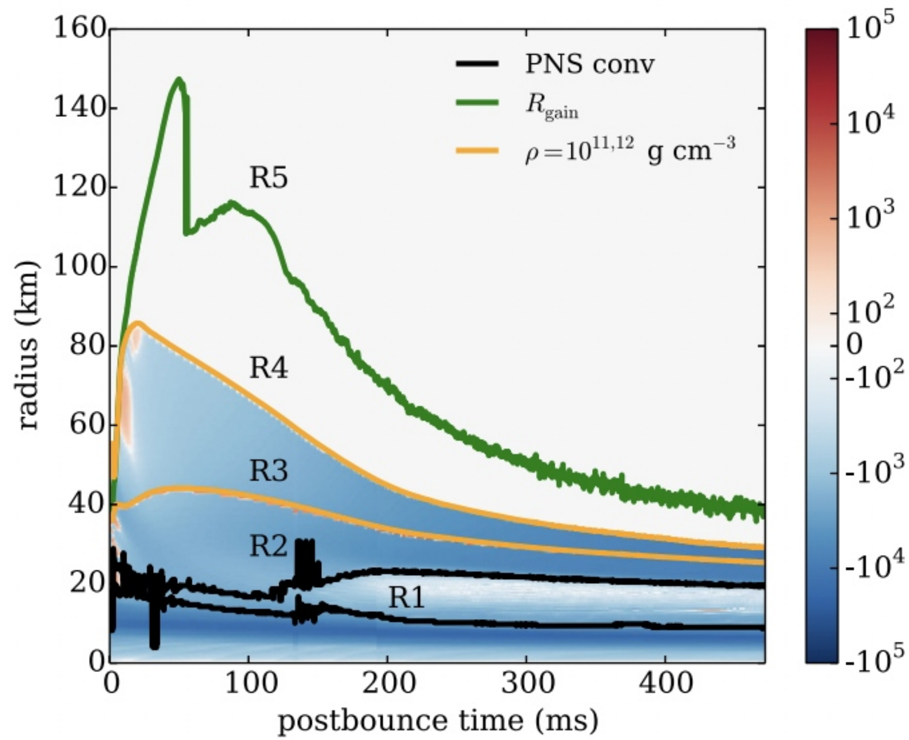
Torres-Forrne et al. '19

Sources of GW Emission

- **Drivers** of mode excitation difficult to establish: Convection in gain region, PNS convection, SASI?
- Various means to establish causality:
  - Temporal correlations of forcing phenomenon and oscillation mode
  - Regional analysis
  - Theoretical considerations on energetics and coupling efficiency (e.g. frequency overlap)
  - Attribution of drivers is important: What does the observed GW power constrain?

Regional Analysis

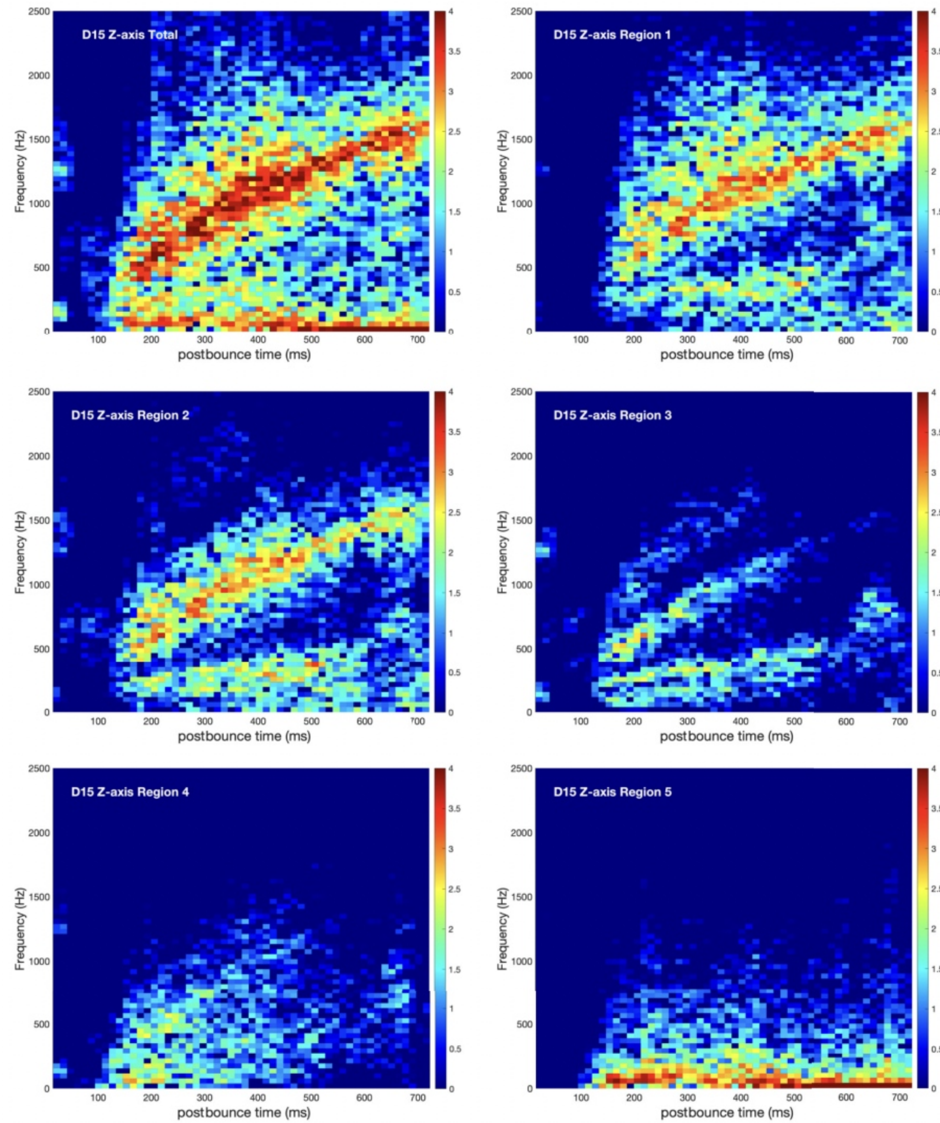
- Integral in quadrupole formula can be split into radial shells (Andresen et al. '17, Mezzacappa et al. '23, Murphy et al. '25)
- Split must be done carefully (Zha et al. '25) and interpretation remains difficult) interference terms
- Regional analysis tends to suggest at least some role for PNS convection for high-frequency emission (Andresen et al. '17, Mezzacappa et al. '23, Murphy et al. '25)



Mezzacappa et al. '23

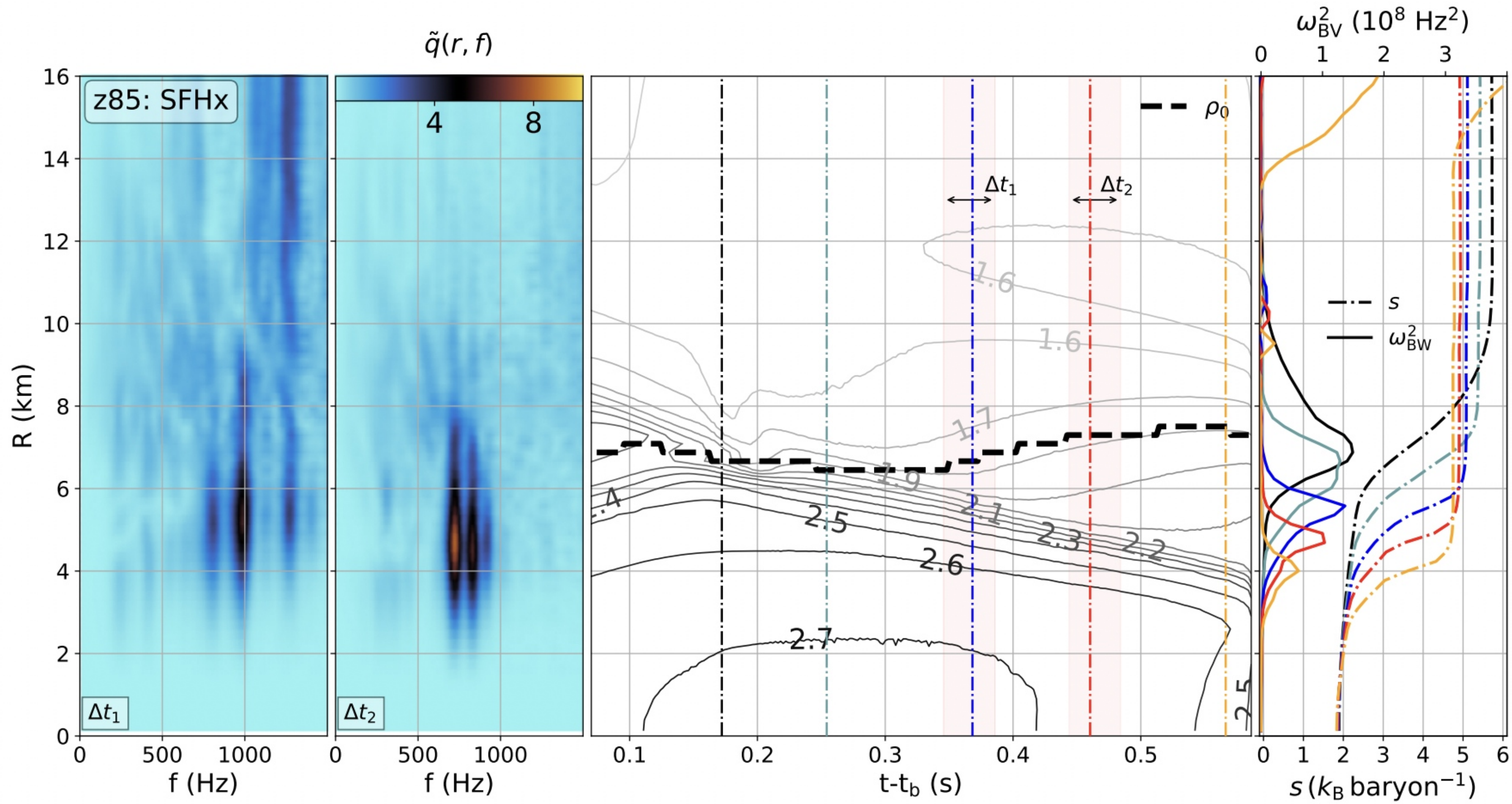
Regional Analysis

- Integral in quadrupole formula can be split into radial shells (Andresen et al. ‘17, Mezzacappa et al. ‘23, Murphy et al. ‘25)
- Split must be done carefully (Zha et al. ‘25) and interpretation remains difficult) interference terms
- Regional analysis tends to suggest at least some role for PNS convection for high-frequency emission (Andresen et al. ‘17, Mezzacappa et al. ‘23, Murphy et al. ‘25)



Mezzacappa et al. ‘23

Refinement of Regional Analysis



For sufficiently well-resolved simulation and with short output intervals, a space-time-frequency analysis is possible to directly identify modes present in simulations (Jakobus et al ‘23) and connect to linear theory

Energetics of Mode Excitation

- G-mode excitation by convection roughly determined by convective luminosity, Mach number and time scale τ:

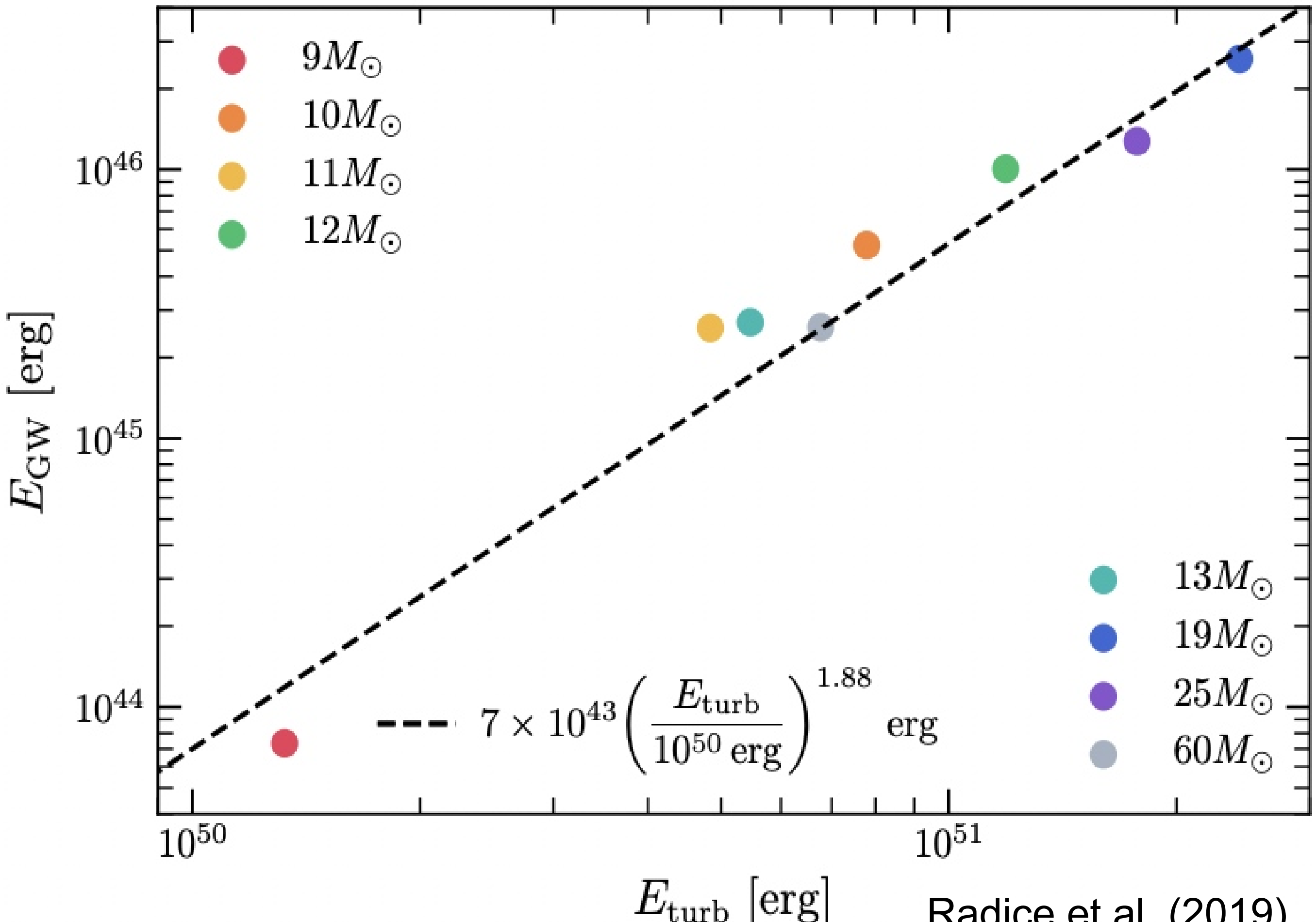
$$E_g \sim \alpha \, \mathrm{Ma} \, L_{\mathrm{conv}} \tau \sim \alpha \, \mathrm{Ma} \, (E_{\mathrm{conv}}/\tau) \tau \sim \alpha \, \mathrm{Ma} \, E_{\mathrm{conv}}$$

- Total GW energy due to excitation by convection in gain region expected to scale as (Powell & Mueller '19):

$$E_{\mathrm{GW}} \sim \frac{4\pi G (f \tau \mathrm{Ma} L_{\mathrm{conv}} T)^2}{c^5} = \frac{4\pi G f^2 \tau^2 \mathrm{Ma}^2 E_{\mathrm{turb}}^2}{c^5}$$
$$= 4.2 \times 10^{43}\,\mathrm{erg} \left(\frac{f}{1000\,\mathrm{Hz}}\right)^2 \left(\frac{\tau}{20\,\mathrm{ms}}\right)^2 \left(\frac{\mathrm{Ma}^2}{0.3}\right) \left(\frac{E_{\mathrm{turb}}}{10^{50}\,\mathrm{erg}}\right)^2$$

- Radice et al. (2019) diagnose such a correlation with the turbulent energy flux in the gain region in their models
- **Issue:** Similar GW energy predicted for excitation by PNS convection (higher turbulent energy, lower Mach number)

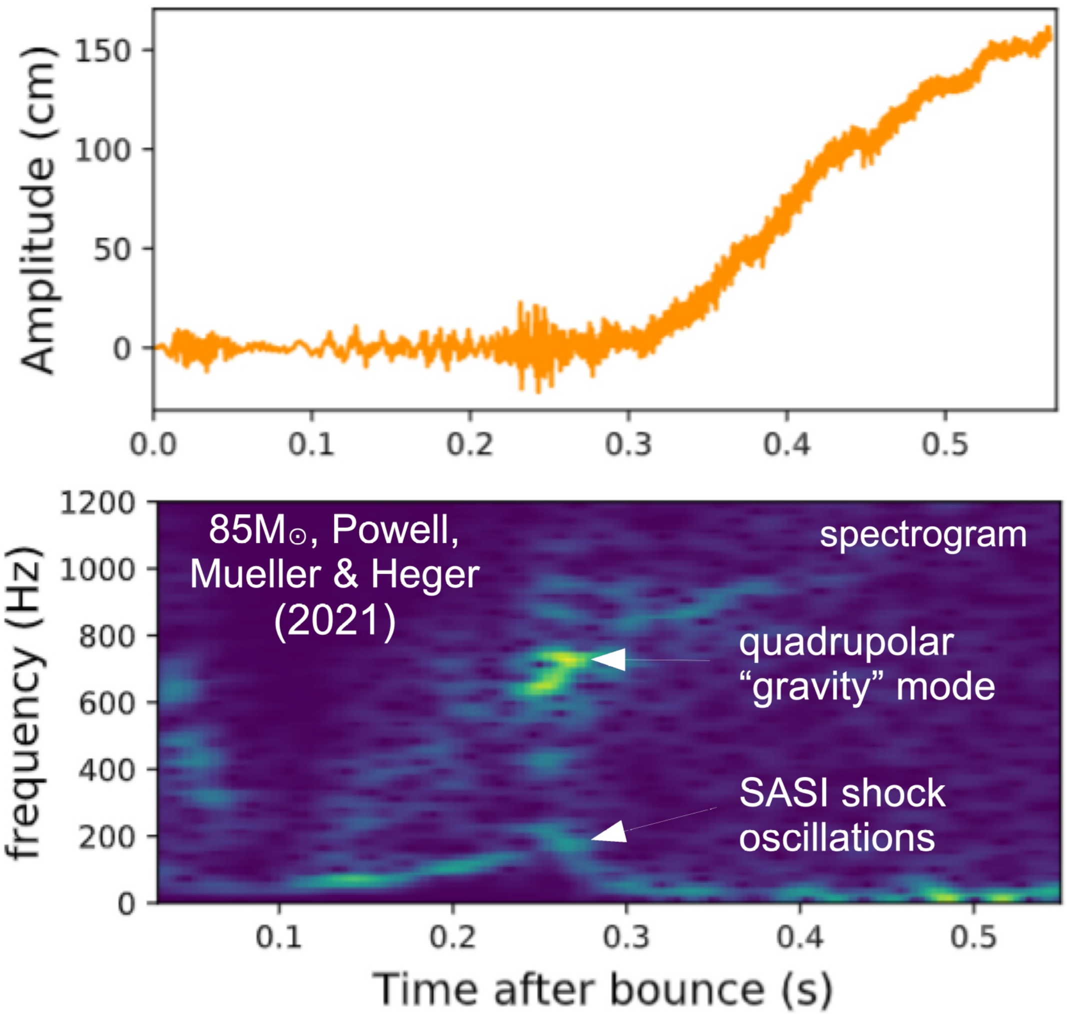
Energetics of Mode Excitation



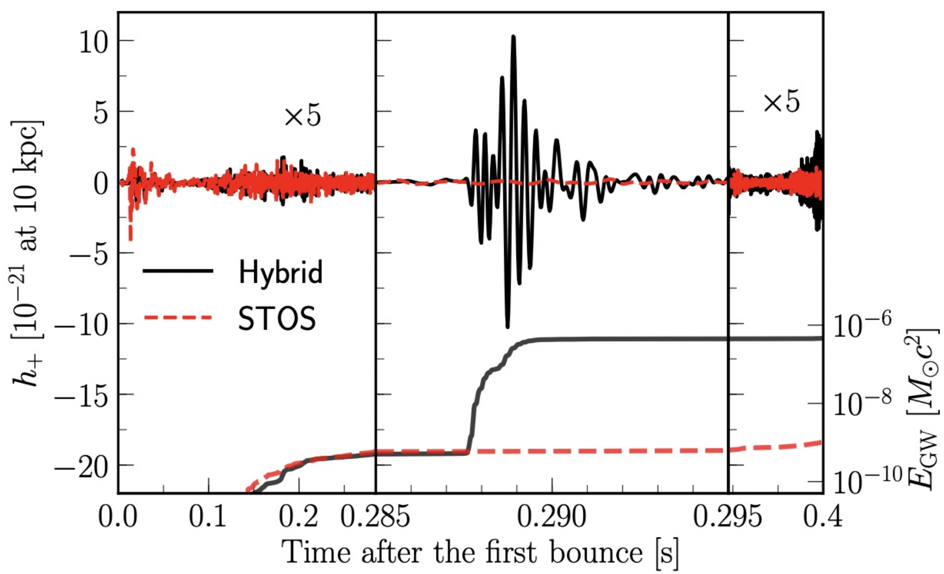
Radice et al. (2019)

Temporal Correlations

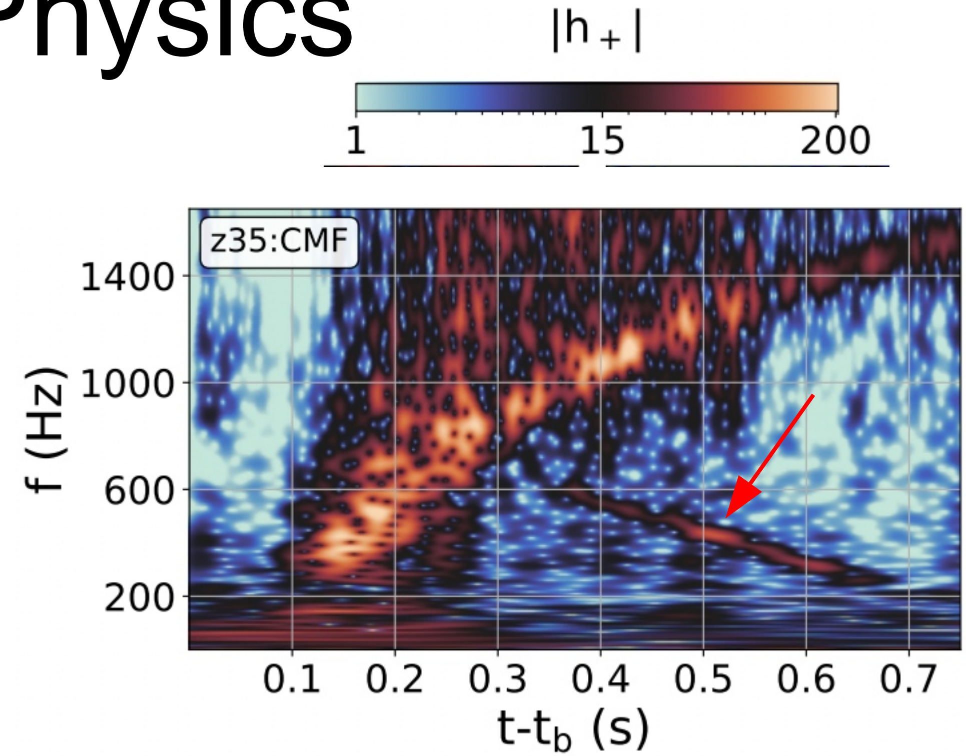
- Emission tends to peak around onset of explosion when the turbulent kinetic energy in the gain region is high
- Emission proceeds when convection in the gain region becomes weak after explosion
- Suggests definite role for PNS convection after explosion & contribution from both gain region and PNS convection earlier
- Attribution remains non-trivial



GW as Probes of High-Density Nuclear Physics



Zha et al. (2020)



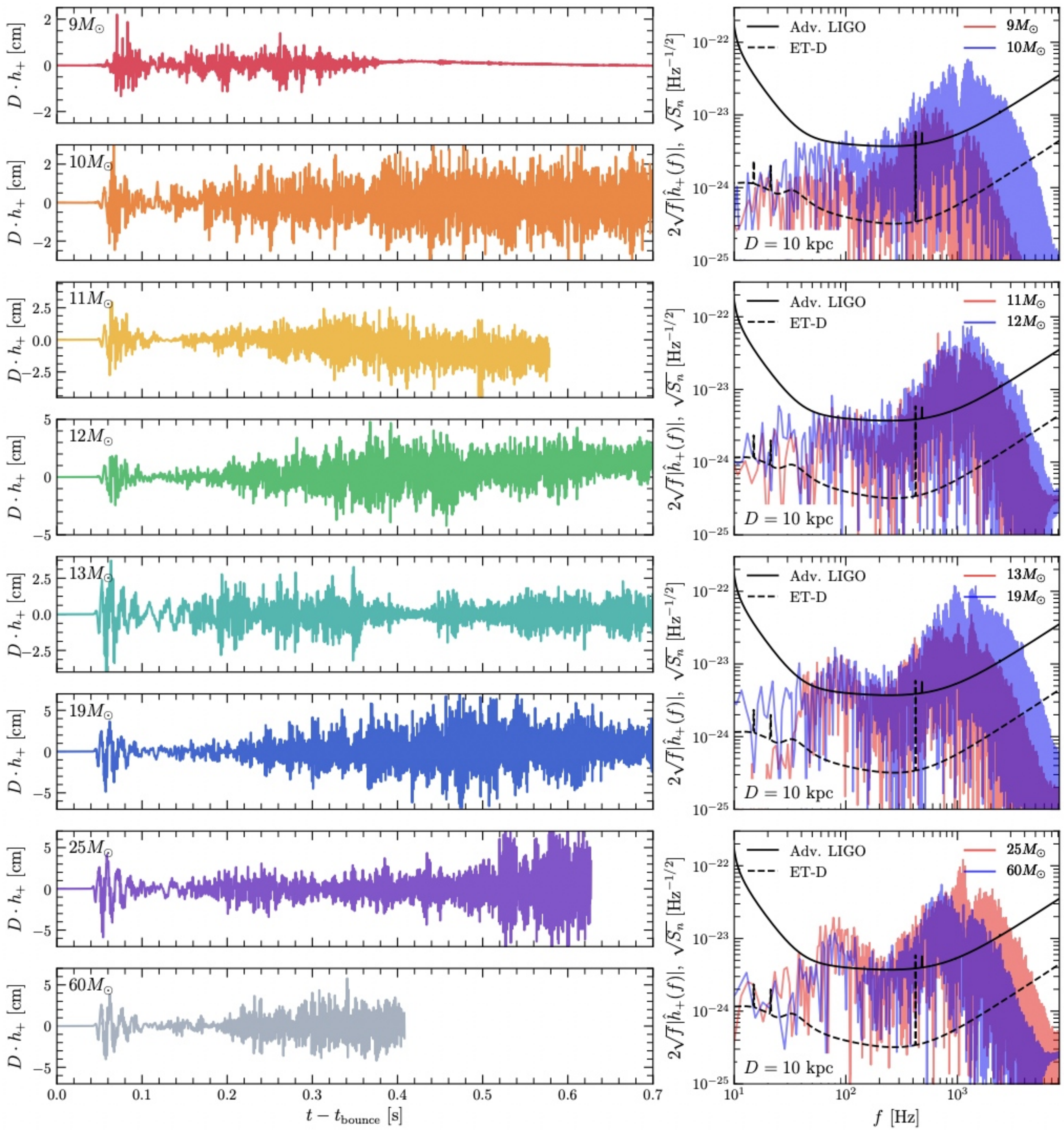
Jakobus et al. (2023)

- Influence of nuclear EoS on dominant band is indirect (neutron star radius)
- GW signal may provide more direct clues to high-density physics
- Strong high-frequency signal may occur after first-order phase transition to quark matter and second collapse and bounce (Zha et al. 2020)
- Core g-mode probes thermodynamic derivatives at the core mantle interface (mass coordinate ~0.6M$_\odot$):

$$\tilde{\omega}_{\rm BV}^{\rm approx,fix} \approx 0.55 \times \sqrt{\frac{1}{\pi} G M_{\rm mode} \alpha_{\rm approx}^5 \frac{1}{c_s^2} \left(\frac{\partial P}{\partial s}\right)_{\rho, Y_e}} \times 11.93\ k_B/M_\odot,$$

Jakobus et al. (2025)

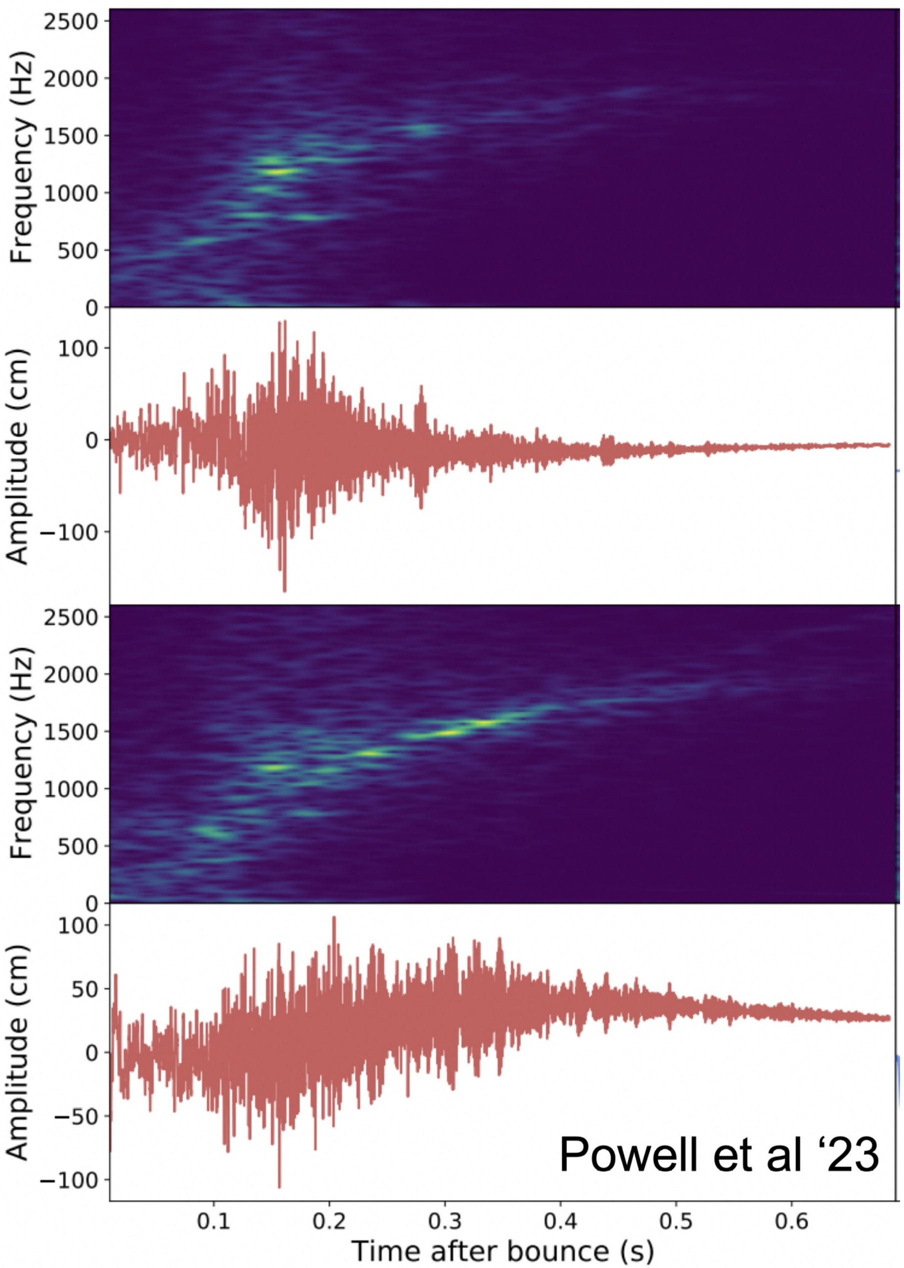
Phenomenology: Progenitor Dependence

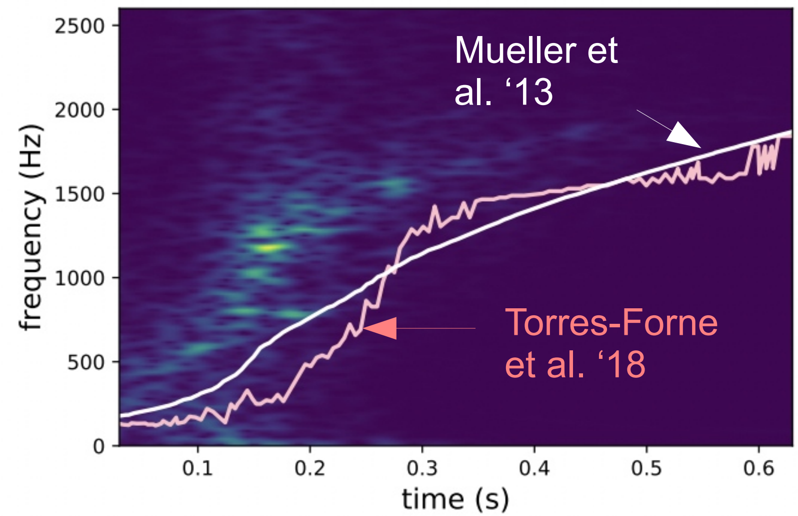


- More massive progenitors (bigger cores) tend to emit more GW energy in 3D (Radice et al. '19)
- Similar trend as in 2D (Mueller et al. '13), but lower amplitudes

Radice et al. (2019)

Impact of Strong Rotation & Magnetic Fields

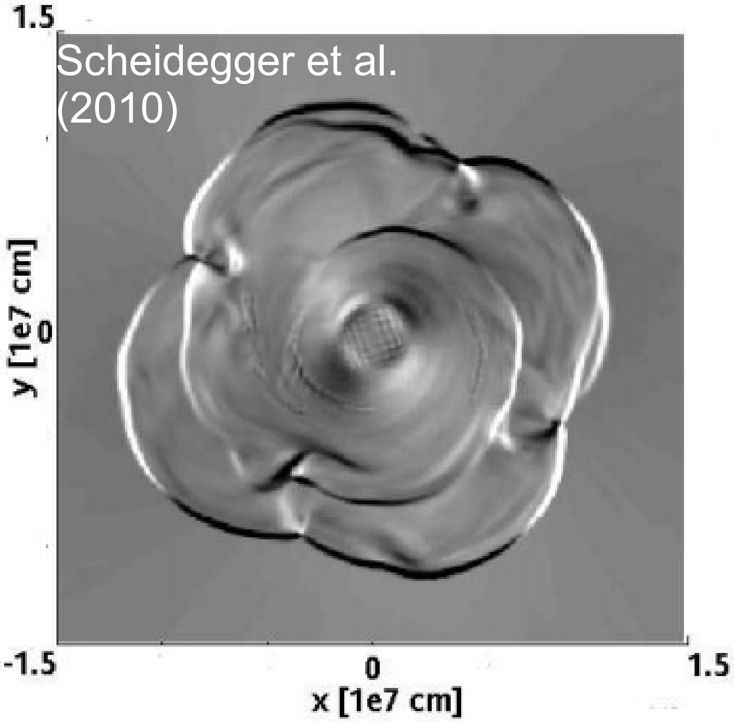


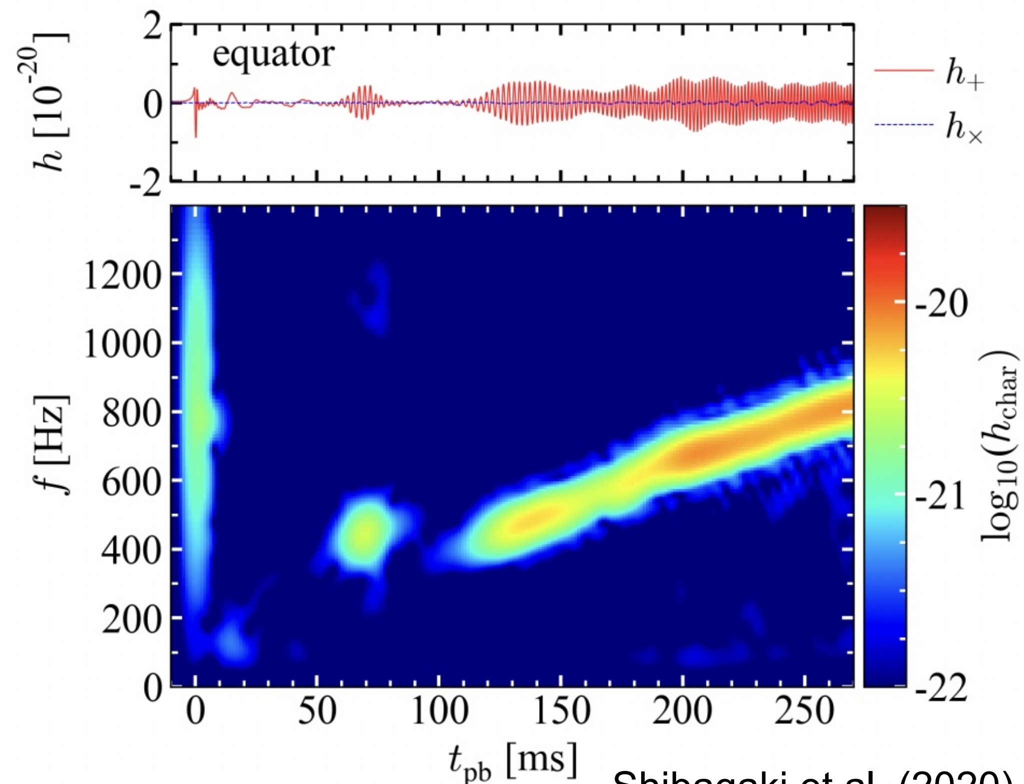


- Qualitatively similar structure for magnetorotational explosions
- But GW amplitudes may be boosted
- Frequency bands can deviate **significantly** from "universal relations"
- Modification of buoyancy frequency by angular momentum gradients:

$$N^2 = N_{\mathrm{BV}}^2 + \frac{1}{\varpi^3}\frac{\partial j^2}{\partial \varpi}\sin\theta,$$

Rotation After Bounce: More Spectacular Signals

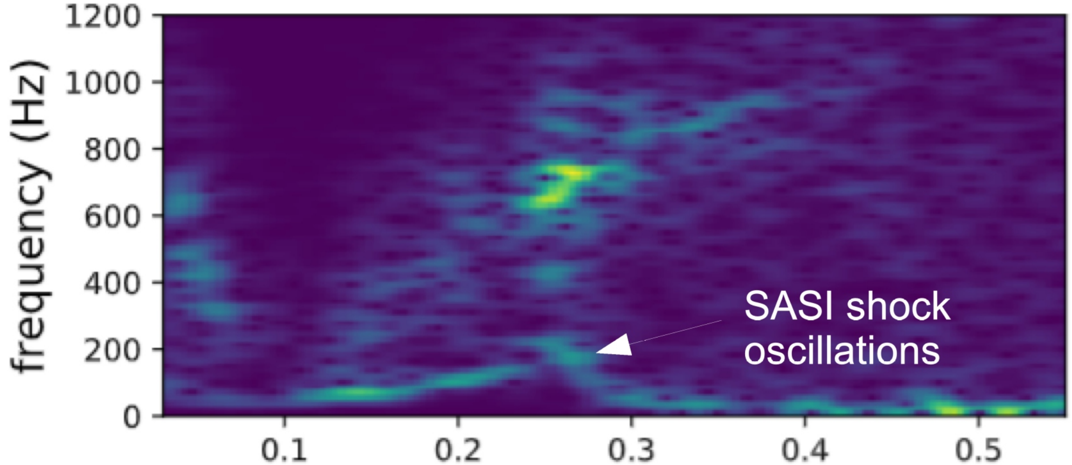


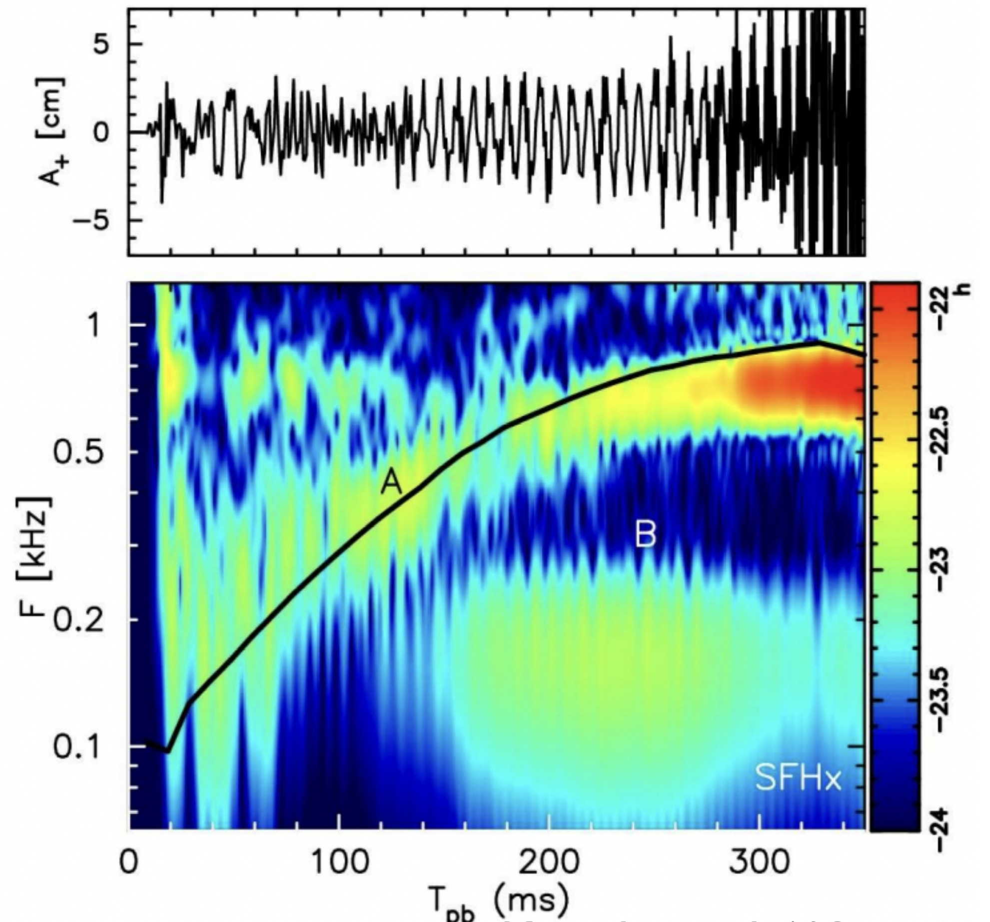


Shibagaki et al. (2020)

- Possibility of triaxial low |T|/W-instability for **fast rotation** around bounce (Scheidegger et al. ‘10)
- Strong & sustained signal from low triaxial instability also seen after bounce (Kuroda et al. 2016, Shibagaki et al. 2020)
- Progenitor rotation remains big caveat

Low-Frequency Emission: SASI





Kuroda et al. '16

- SASI can contribute signal in the 100-300Hz region
- Signal often short & intermittent
- Some cases with stable SASI signal reported, e.g. in non-exploding models (Kuroda et al. '16) and very massive progenitors
- Frequency structure needs to be better understood, may involve doubling over base frequency

$$T_{\mathrm{SASI}} = 19\,\mathrm{ms}\left(\frac{r_{\mathrm{sh}}}{100\,\mathrm{km}}\right)^{3/2} \ln\left(\frac{r_{\mathrm{sh}}}{r_{\mathrm{PNS}}}\right)$$

(Mueller & Janka '14)

Very Low-frequency Emission

- Signal components <10 Hz may become relevant for some future detectors
- Two phenomena contribute to very slowly varying amplitudes:
  - Anisotropic neutrino emission (Epstein 1978)

$$h_{ij}^{\mathrm{TT}}(\mathbf{X},t) = \frac{4G}{c^4 R}\int_{-\infty}^{t-R/c} dt' \int_{4\pi} d\Omega' \frac{(n_i n_j)^{\mathrm{TT}}}{1-\cos\theta} \cdot \frac{dL_\nu(\mathbf{\Omega}',t')}{d\Omega'}$$

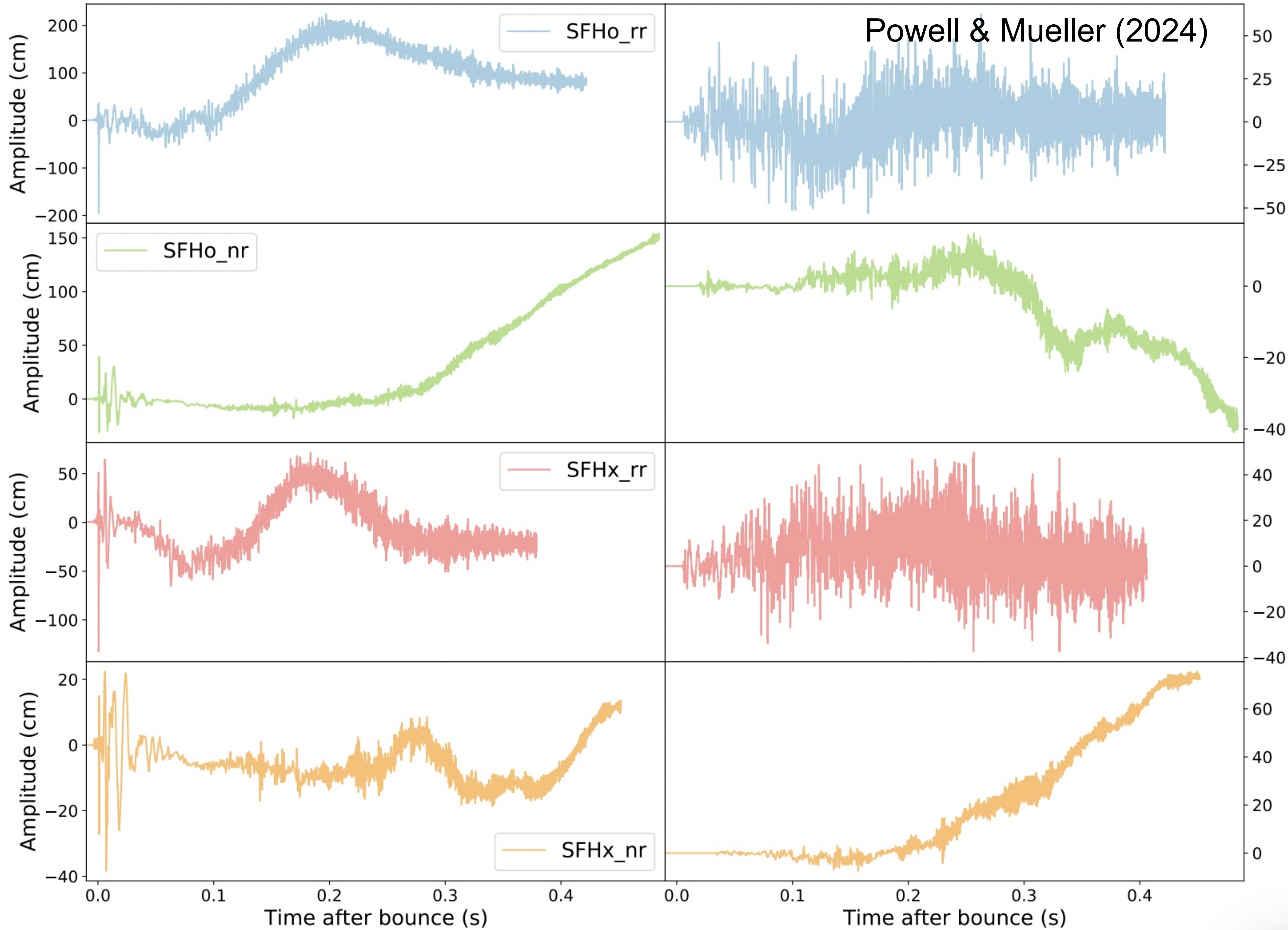
  - Asymmetric shock propagation (likely subdominant)

$$A_{20,\mathrm{shock}}^{\mathrm{E2}} \approx \frac{256\pi^{3/2}}{5\sqrt{15}}\bar{\rho}_{\mathrm{p}}\,(\beta-1)\,a_0^3\left[(4+\gamma)\,a_2\dot{a}_0 + \dot{a}_2 a_0\right].$$

Pre-shock density

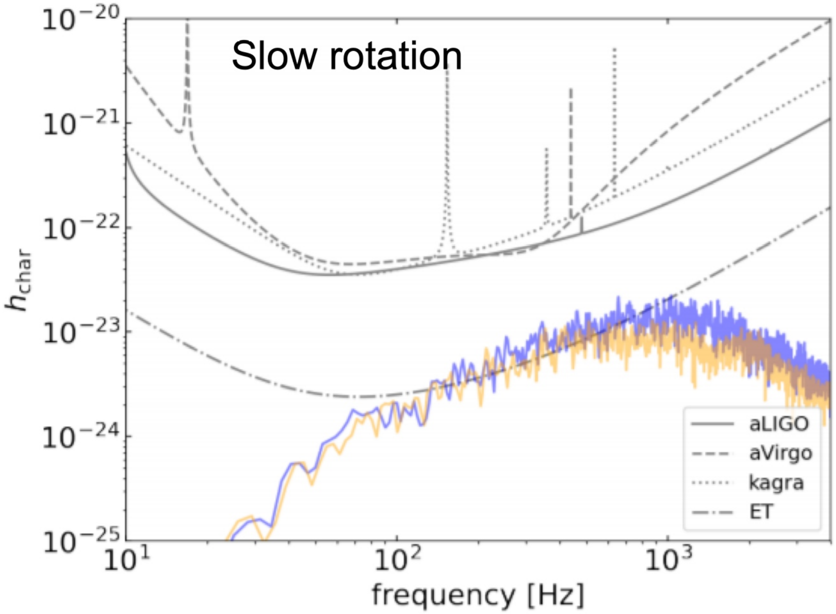
Compression ratio

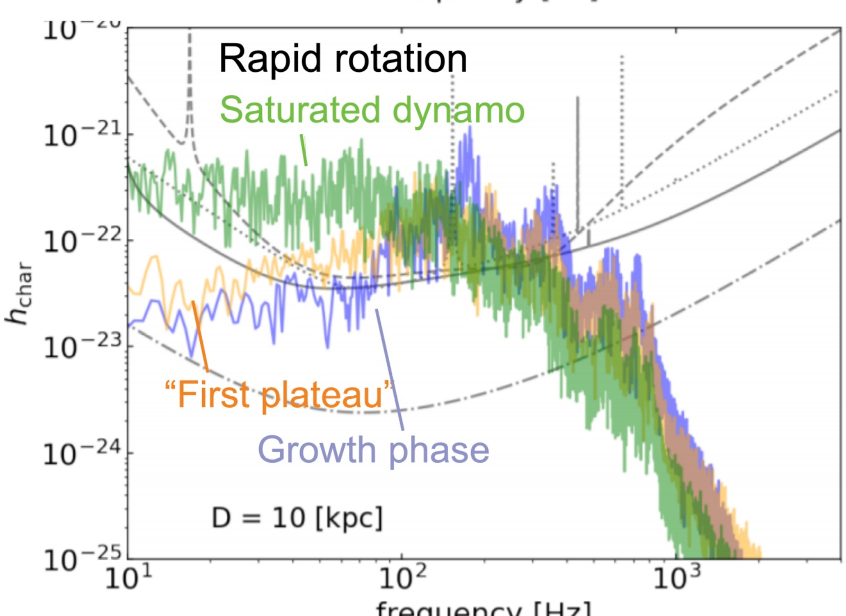
Monopole and quadrupole coefficients of shock radius

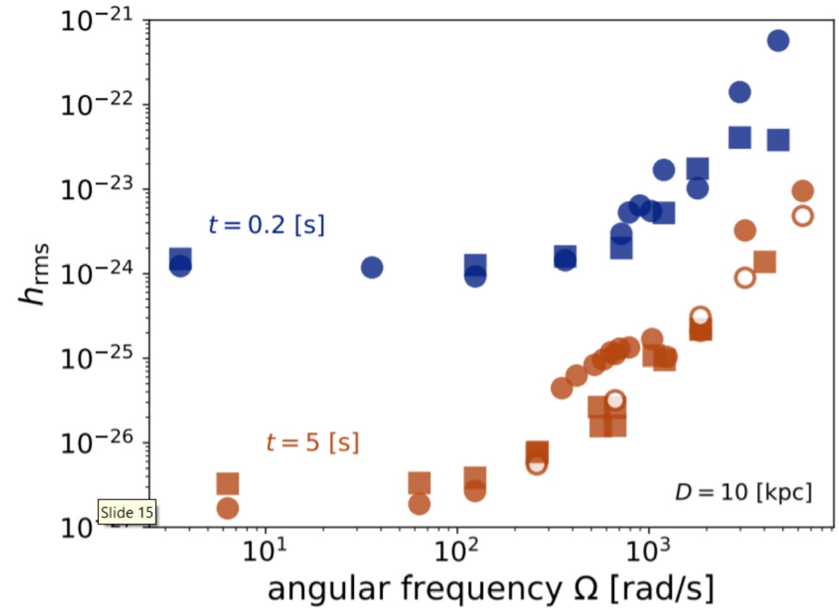


Very low-frequency component requires further investigation (using second-long simulations) ← Extrapolation/cut-off problem

Late-time GW Emission from Proto-Neutron Star Convection







Raynaud, Cerdá-Durán & Guilet ('22)

Raynaud et al. ('20): anelastic long-time MHD simulations of PNS convection

- Strongly enhanced GW emission for rapid rotation (Rossby number < 1)
- Low-frequency excess as "smoking gun" of strong dynamo
- Interpreted as magnetically modified inertial mode

Lots of Potential – One Caveat

- The predicted gravitational wave signal is a very sensitive probe of the simulated dynamics of a supernova.
- It can be contaminated by numerical instabilities and analysis artefacts.
- (New) practitioners need to be aware of these risks.

Numerical Issues with GW Extraction

- GWs extracted using quadrupole formula (either stress formula or time-integrated quadrupole formula):

$$h_{ij} D = \frac{2G}{c^4} \mathrm{STF} \int \rho v_i v_j - x_i \partial_j \Phi \, \mathrm{d}V \quad \text{or} \quad h_{ij} D = \frac{2G}{c^4} \frac{\partial}{\partial t} \mathrm{STF} \int x_i \rho v_j \, \mathrm{d}V$$

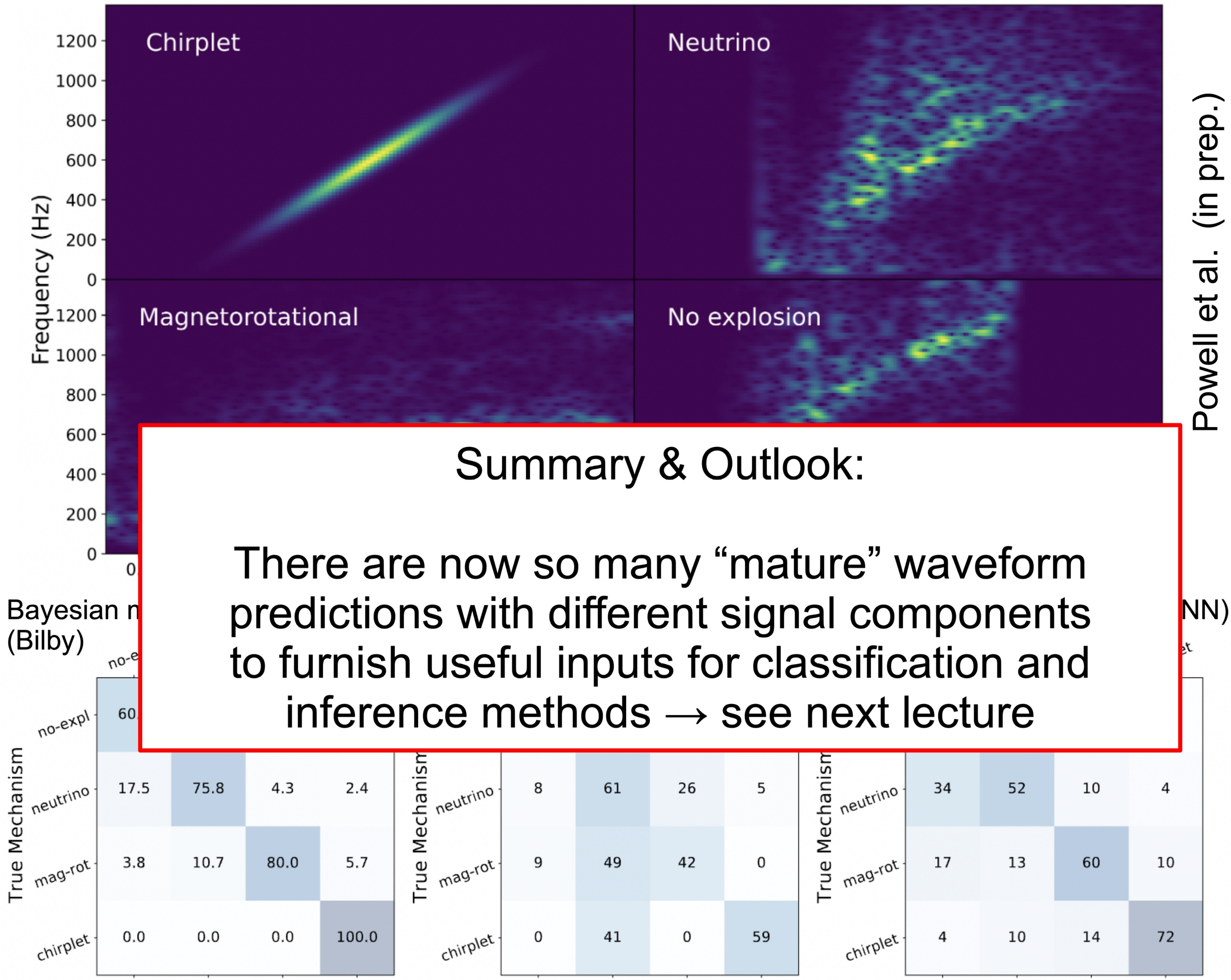
- This involves a projection onto l=2 tensor spherical harmonics.
- For a radial flow field, the amplitude should be zero because

  $\mathrm{STF} \int x_i x_j \, \mathrm{d}\Omega = 0$ (no overlap between l=0 and l=2)

- However, for a discrete grid, one generally has

$$\mathrm{STF} \sum_{\text{sphere}} x_i x_j \Delta \Omega \neq 0$$

- Thus the l=0 component of the flow can pollute the GW signal.
- This can be a serious problem at low resolution (>3°).
- On spherical polar grids, one can project out the l=0 component before applying the quadrupole formula.

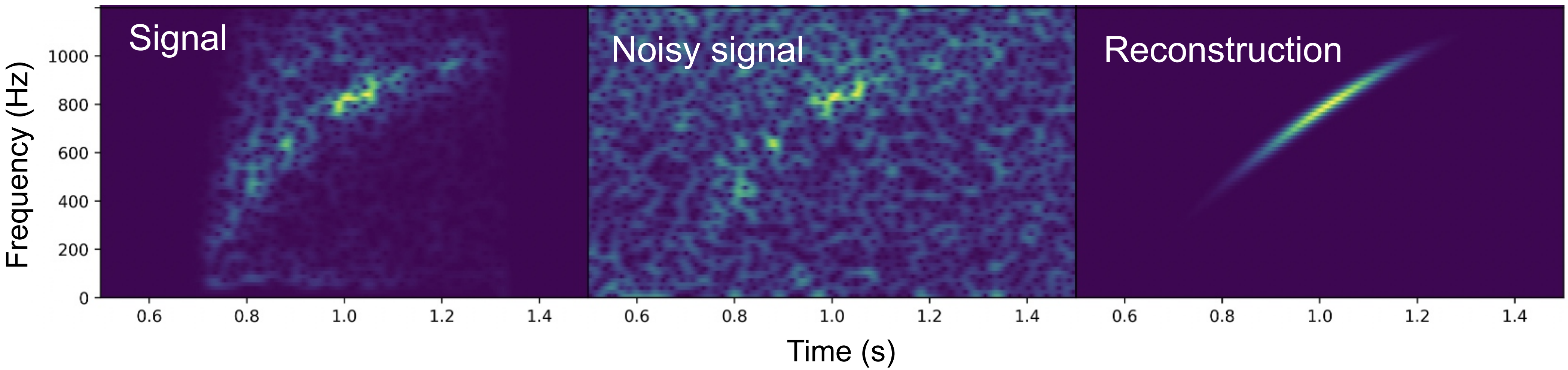


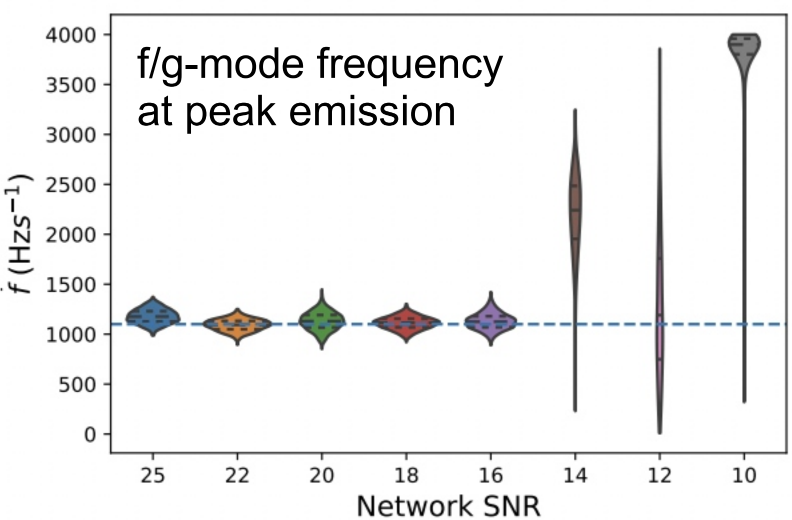
Powell et al. (in prep.)

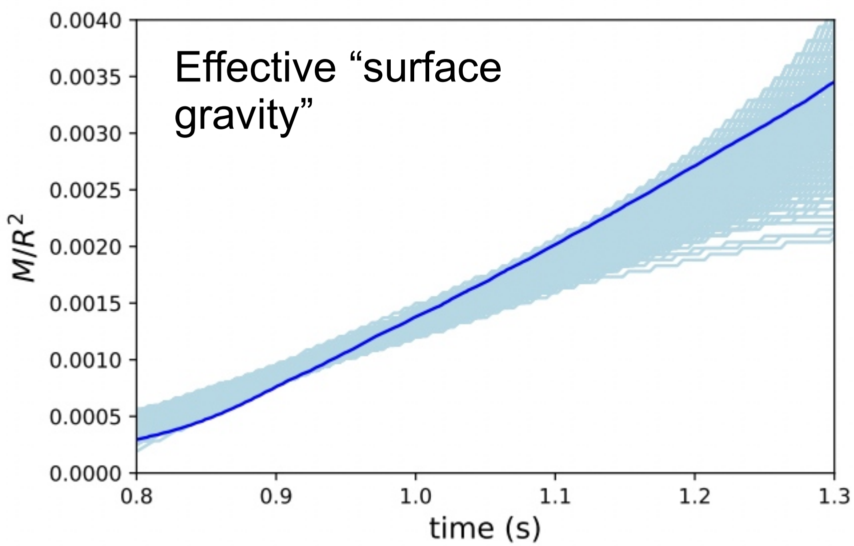
Summary & Outlook:

There are now so many “mature” waveform predictions with different signal components to furnish useful inputs for classification and inference methods → see next lecture

Outlook: Inference of Physical Parameters







Inference of GW waveform, explosion, and neutron star parameters with *Bilby*

Powell & Mueller (2022)

Thank you for your attention!