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Core-Collapse Supernova Simulations with a Multidimensional Full Boltzmann Solver in `Gmunu`

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Neutrinos in simulations

- Neutrinos play a crucial role in CCSNe $\longrightarrow$ included in simulations
- Boltzmann equation is $3+3+1$D ($\mathbf{x}$, $\mathbf{p}$ and t)

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{x}} f + \mathbf{F} \cdot \nabla_{\mathbf{p}} f = \left(\frac{\partial f}{\partial t}\right)_{\text{collisions}}$$

Cardall et al 2013

- Approximations (e.g. moment-based, ray-tracing) comp efficient
- What is lost through these approximations ? Effect on system ? On emitted signals ?

$\Longrightarrow$ Solve the Boltzmann equation fully

General-relativistic Boltzmann equation

The change in f along the geodesic is equal to the invariant collision integral $C[f]$ (interactions)

$$\frac{df}{d\lambda} = C[f] \tag{1}$$

Requires explict choices

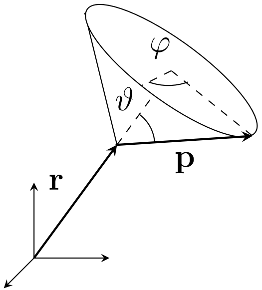
- Phase-space coordinate system
- Reference frame in which momenta are defined

$\Rightarrow$ Significantly impacts accuracy, efficiency, treatment of neutrino-matter coupling

Momentum space coordinates

Complement space with momentum in spherical coordinates

$$p^{\hat{1}} = \varepsilon \cos\vartheta$$
$$p^{\hat{2}} = \varepsilon \sin\vartheta \cos\varphi$$
$$p^{\hat{3}} = \varepsilon \sin\vartheta \sin\varphi$$



Choice of reference frame?

- Local orthonormal frame (*e.g.*, Akaho et al 2021)
- Local comoving frame (this work)

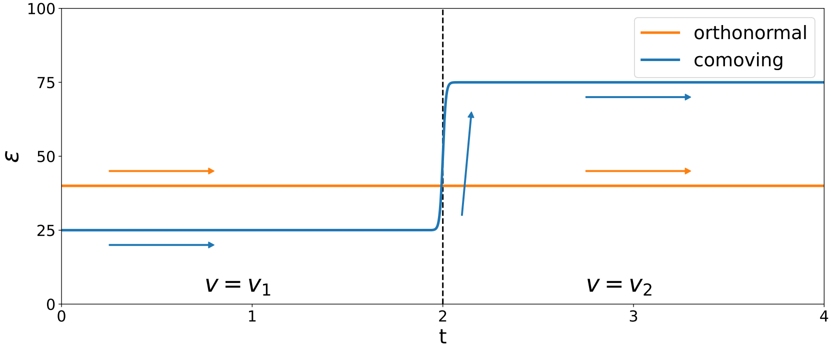
Orthonormal vs Comoving frame

Orthonormal frame

- Easier momentum advection

Comoving frame

- Tricky momentum advection



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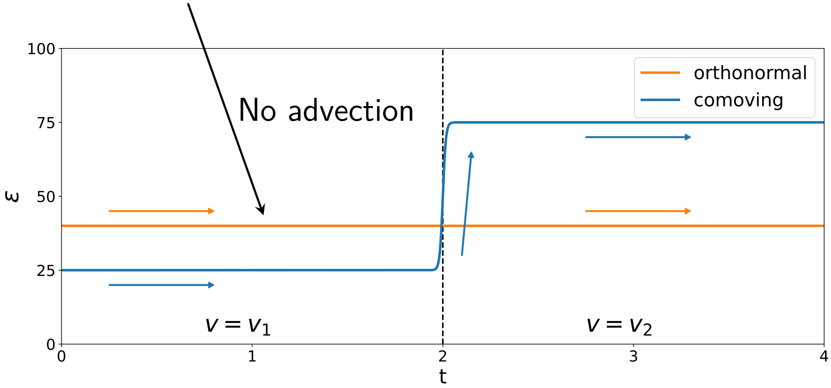

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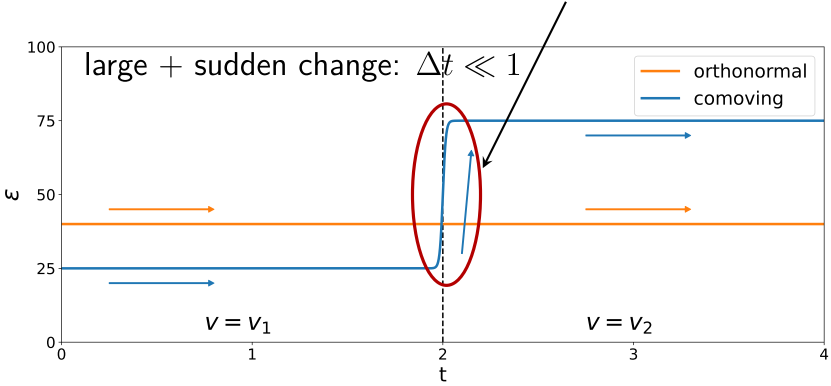


Orthonormal vs Comoving frame

Orthonormal frame

- Easier momentum advection (fluid-independent)
- Mapping frames for interaction terms

Comoving frame

- Tricky momentum advection (fluid-dependent)
- Straightforward interaction terms

computed in **comoving frame**

$$C[f] \implies \text{opacities, interaction rates,}\ldots$$

Conservative formulation of the Boltzmann equation

Conservative form for ν number + curvilinear coordinates $p^{\tilde{\imath}}$

$$\partial_t q + \frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial x^i}\left(\sqrt{\gamma}\, F^i\right) + \frac{p^{\hat{0}}}{\sqrt{\lambda}} \frac{\partial}{\partial p^{\tilde{\imath}}}\left(\frac{\sqrt{\lambda}}{p^{\hat{0}}} \mathcal{F}^{\tilde{\imath}}\right) = s_{\text{rad}}$$

3-metrics: γ_{ij} (position space), $\lambda_{\tilde{\imath}\tilde{\jmath}}$ (momentum space) (*Cardall et al 2013*)

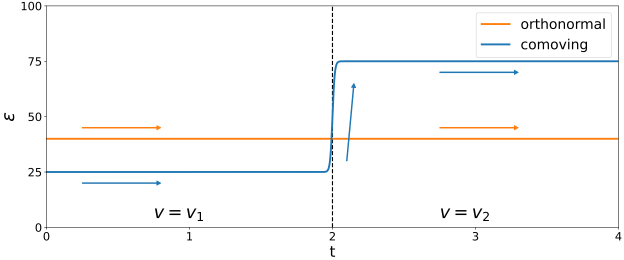
$\Longrightarrow (1+6)D$ conservation equation

$\Longrightarrow$ `Gmunu` extended for $6D$ finite volume method
`Gmunu`: GRMHD code with advanced microphysics

From spacetime to phase-space

Some generalization and extra care required

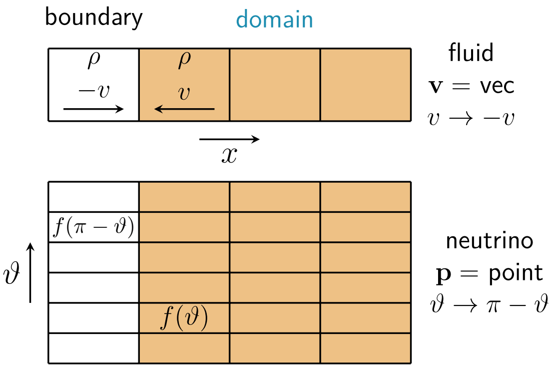
- Coordinate systems + reference frame**s**



From spacetime to phase-space

Some generalization and extra care required

- Coordinate systems + reference frames
- Boundary conditions



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From spacetime to phase-space

Some generalization and extra care required

- Coordinate systems + reference frames
- Boundary conditions
- Flux discretization

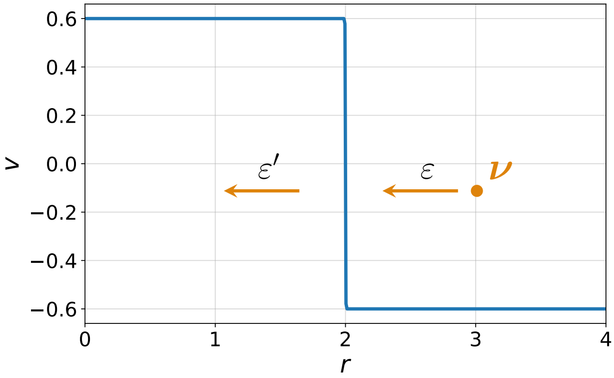
$$f \nabla_{\mu} L^{\mu}{}_{\hat{\mu}} p^{\hat{\mu}} = f p^{\hat{0}} \mathcal{D}_{\tilde{\imath}} \left(\frac{1}{p^{\hat{0}}} P^{\tilde{\imath}}_{\hat{\imath}} \Gamma^{\hat{\imath}}{}_{\hat{\mu}\hat{\nu}} p^{\hat{\mu}} p^{\hat{\nu}} \right)$$

Must ensure that this is true in the discrete limit

Numerical test - velocity gradient

Comoving frame $\Longrightarrow$ How sensitive to velocity gradients?

- Velocity gradients specific concern in comoving frame
- Discretized equations sensitive to local fluid velocity profile

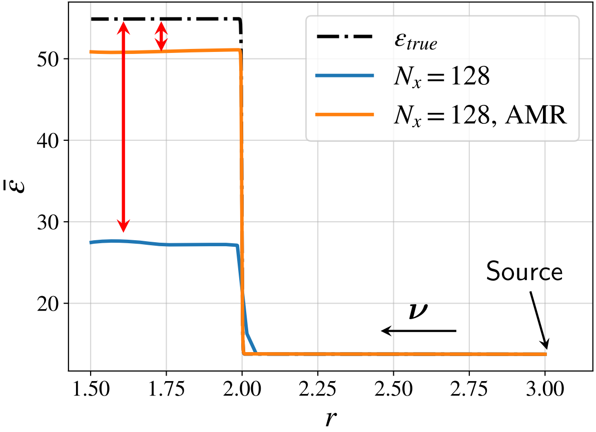


Numerical test - velocity gradient

Mean energy trajectory

$$\bar{\varepsilon}(r) = \frac{\sum_i f(r, \varepsilon_i)\varepsilon_i \Delta\varepsilon}{\sum_i f(r, \varepsilon_i)\Delta\varepsilon}$$

- Velocity gradients must be accurately computed
- Adaptive mesh refinement (AMR) important
- Extreme case, smaller shocks in CCSNe



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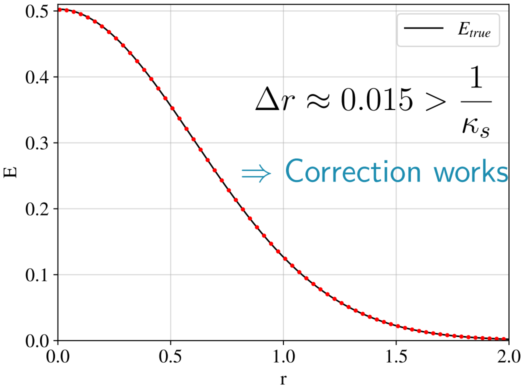
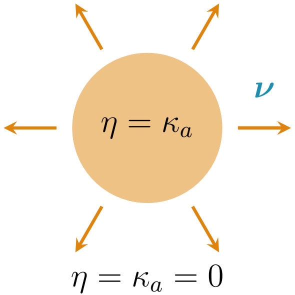
Numerical test - diffusive limit

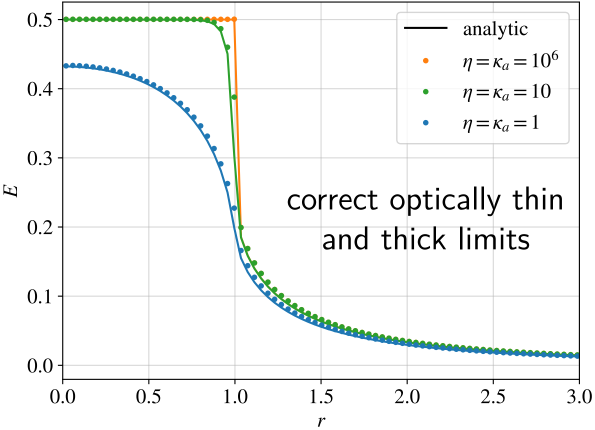
- In dense environments neutrino transport can become diffusive
- Correction when mean free path not resolved (*e.g.*, Mezzacappa & Bruenn 1993)
- Large scattering opacity $\kappa_s = 10^3$



Numerical test - homogeneous radiating sphere

- Toy model for hot neutron star





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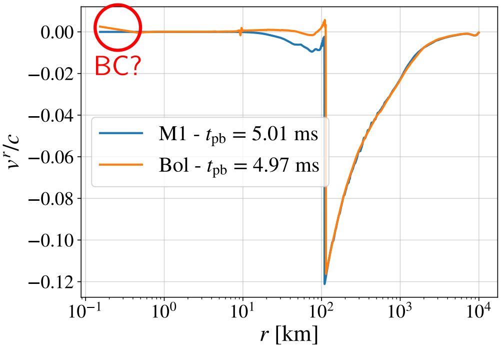
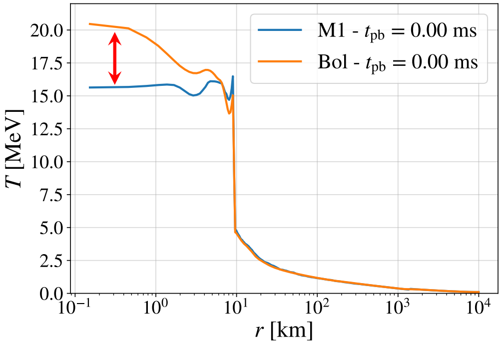
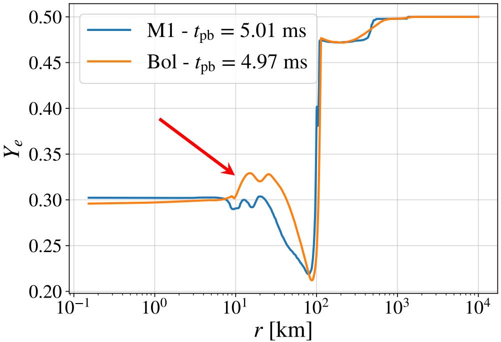
Core-collapse supernovae

- $15 M_\odot$ progenitor, SFHo EOS
- `Weakhub` interaction table (Ng et al 2024)
- $N_\varepsilon = 20,\ N_\vartheta = 10$
- Collapse + bounce + expanding shock
- A couple of unexpected behaviours



Core-collapse supernovae





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Concluding Remarks

- Multidimensional relativistic full Boltzmann solver in `Gmunu`
  - Comoving frame
  - Non-trivial discretisation of phase-space
  - Accurate estimate of velocity gradients required
- Good agreement in benchmarking tests
  - Tests for momentum advection + interaction with a fluid
  - Simulations of core-collapse supernova to be improved
- Paper with full details and tests in prep.
- Tier-1 Supercomputing time (8M CPUh) for 2D simulations

Appendix

The Boltzmann equation in Gmunu

- Reference metric $\hat{\gamma}_{ij}$ + conformal metric $\gamma_{ij} = \psi^4(\mathbf{x})\,\bar{\gamma}_{ij}$

$$\partial_t q + \frac{1}{\sqrt{\hat{\gamma}}}\frac{\partial}{\partial x^i}\left(\sqrt{\hat{\gamma}}\,F^i\right) + \frac{p^{\hat{0}}}{\sqrt{\lambda}}\frac{\partial}{\partial p^{\tilde{i}}}\left(\frac{\sqrt{\lambda}}{p^{\hat{0}}}\,\mathcal{F}^{\tilde{i}}\right) = s_{\mathrm{rad}}$$

$$q = \psi^6\sqrt{\frac{\bar{\gamma}}{\hat{\gamma}}}\mathcal{L}_{\hat{\mu}}p^{\hat{\mu}}\,f \qquad F^i = \psi^6\sqrt{\frac{\bar{\gamma}}{\hat{\gamma}}}\left(\alpha\,\ell^i{}_{\hat{\mu}} - \beta^i\mathcal{L}_{\hat{\mu}}\right)p^{\hat{\mu}}\,f$$

$$\mathcal{F}^{\tilde{i}} = -\psi^6\sqrt{\frac{\bar{\gamma}}{\hat{\gamma}}}\alpha\,P^{\tilde{i}}{}_{\hat{\imath}}\,\Gamma^{\hat{\imath}}_{\hat{\mu}\hat{\nu}}\,p^{\hat{\mu}}\,p^{\hat{\nu}}\,f \qquad s_{\mathrm{rad},s} = \psi^6\sqrt{\frac{\bar{\gamma}}{\hat{\gamma}}}\,\alpha\,S_{\mathrm{rad}}$$

$$L^{\mu}{}_{\hat{\mu}} = n^{\mu}\mathcal{L}_{\hat{\mu}} + \ell^{\mu}{}_{\hat{\mu}}$$

Riemann solver - characteristic speeds

HRSC requires characteristic speeds of the system of equation
Characteristic speeds c_x^i and $c_p^{\tilde{i}}$ easily obtained

$$F^i = G^i\, q \qquad \mathcal{F}^{\tilde{i}} = \mathcal{G}^{\tilde{i}}\, q$$

$$c_x^i = \frac{\partial F^i}{\partial q} = G^i \qquad c_p^{\tilde{i}} = \frac{\partial \mathcal{F}^{\tilde{i}}}{\partial q} = \mathcal{G}^{\tilde{i}}$$

No discontinuity in characteristic speeds in momentum space

$$c_p^{\tilde{i}} = -\frac{1}{\mathcal{L}_{\hat{\mu}} p_s^{\hat{\mu}}} \alpha\, P^{\tilde{i}}{}_{\hat{\imath}}\, \Gamma^{\hat{\imath}}{}_{\hat{\mu}\hat{\nu}}\, p_s^{\hat{\mu}}\, p_s^{\hat{\nu}}$$

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Scattering kernels

Elastic scattering

$$C_{ES}[f] = p^{\hat{0}}\varepsilon^2 \int d\Omega'_p \, R_{\mathsf{ES}}(\varepsilon,\omega)\Big[f(\varepsilon,\Omega'_p) - f(\varepsilon,\Omega_p)\Big]$$

Legendre expansion of R

$$R_{\mathrm{ES}}\,(\varepsilon,\omega) \approx R_{\mathrm{ES},0}(\varepsilon) + R_{\mathrm{ES},1}(\varepsilon)\omega = \frac{1}{2}\Phi_{\mathrm{ES},0}(\varepsilon) + \frac{3}{2}\Phi_{\mathrm{ES},1}(\varepsilon)\omega$$

$$\kappa_s(\varepsilon) = 4\pi\varepsilon^2\left[R_{\mathrm{ES},0}(\varepsilon) - \frac{1}{3}R_{\mathrm{ES},1}(\varepsilon)\right]$$

M1 only needs κ_s, but we need Φ_0 and Φ_1! $\Longrightarrow$ Weakhub

Diffusive limit

When $\kappa_s \gg 1$, Boltzmann equation $\sim$ diffusion equation.
Ensure this limit is correct with (*Mezzacappa & Bruenn 1993*)

$$\delta^i = 1 - \frac{1}{2}\frac{\Delta x^i \bar{\kappa}}{1 + \Delta x^i \bar{\kappa}},$$

and a κ_s dependent flux reconstruction:

- $c_p^a > 0$: $F^a_{i+1/2,j,k} = \delta^a_{i,j,k} F^a_{i,j,k} + (1 - \delta^a_{i,j,k}) F^a_{i+1,j,k}$;
- $c_p^a < 0$: $F^a_{i+1/2,j,k} = \delta^a_{i+1,j,k} F^a_{i,j,k} - (1 - \delta^a_{i,j,k}) F^a_{i,j,k}$

Caveat: to be checked and generalised to use of Riemann solver...

Implicit evolution

Solving $\mathbf{q}^{n+1} = \mathbf{q}^n + \Delta t\, \mathbf{s}_{\text{rad}}(\mathbf{q}^{n+1})$, which can be rewritten:

$$\mathbf{h} \equiv \mathbf{f}^{n+1} + \left(\mathbf{A}\mathbf{f}^{n+1}\right) \odot \mathbf{f}^{n+1} + \mathbf{B}\mathbf{f}^{n+1} + \mathbf{c} = 0,$$

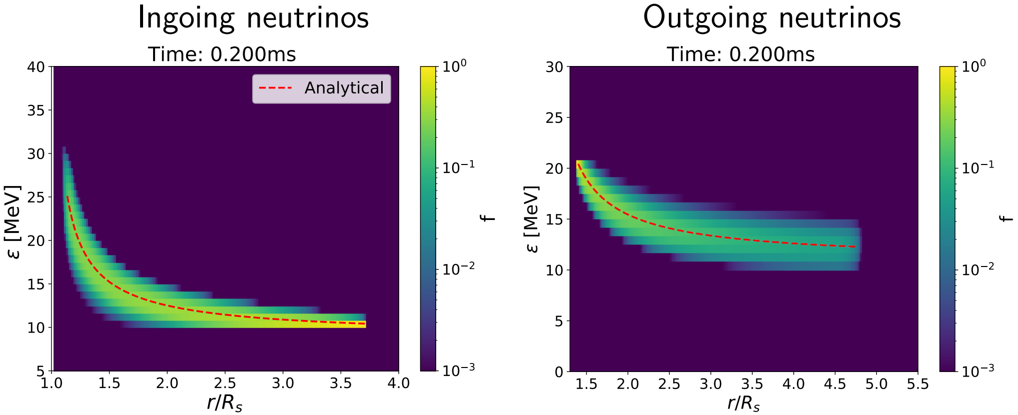
where we assume $L^{n+1} = L^n$.

- Complexity of evalutation: $O(N_{spec}^2 N_p^2)$
- Newton-Raphson iteration:
  - Straighforward, analytical Jacobian
  - Solving iteration (LU decomposition): $O(N_{spec}^3 N_p^3)$
  - If small interaction or Δt is small $\Longrightarrow$ Jacobi method $O(N_{spec}^2 N_p^2)$ (also for M1)

$$(\mathbf{\Delta f})^{(k+1)} = -\mathbf{D}^{-1}\left((\mathbf{J} - \mathbf{D})(\mathbf{\Delta f})^{(k)}\right) - \mathbf{D}^{-1}\mathbf{h}$$

Energy advection in Schwarzschild spacetime

(Akaho et al 2021)



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Angular advection in Schwarzschild spacetime

(Akaho et al 2021)