IMPERIAL

Constrained instantons in scalar field theory

KINGA GAWRYCH

BASED ON 2510.XXXXX WITH BEN ELDER AND ARTTU RAJANTIE

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Motivation

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- **Vacuum decay – very important (electroweak vacuum stability, phase transitions)**

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IN THIS TALK: a method for computing the vacuum decay rate in theories with no instanton solutions

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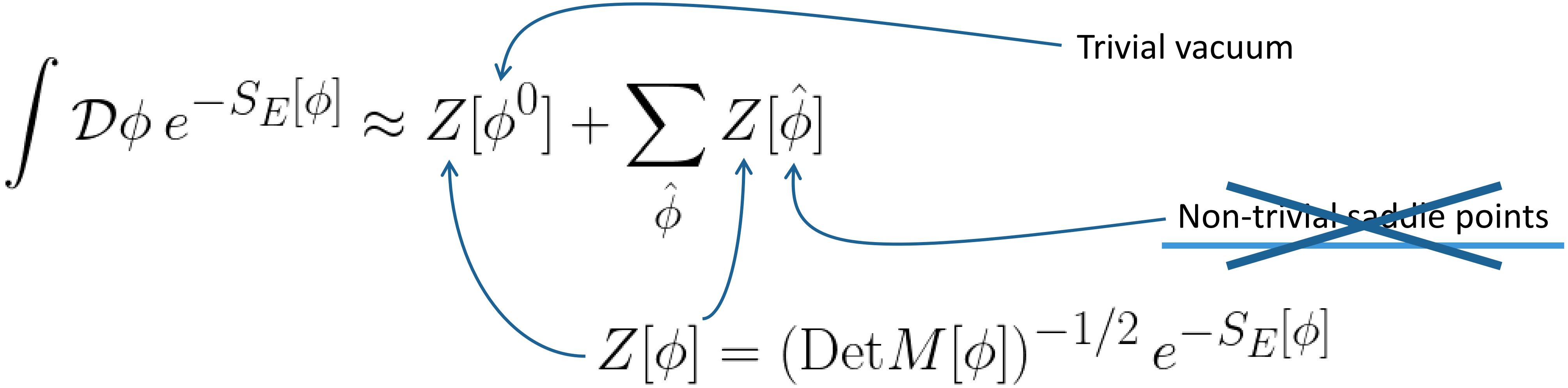

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No saddle points of the action in the space of all functions

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SOLUTION

Split the path integral into sectors, such that in a given sector a saddle point exists + sum over the sectors

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$$\int \mathcal{D}\phi \, e^{-S_E[\phi]}$$

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CAUTION!

$$\mathrm{Det}_{\bar{\xi}}\text{``}\neq\text{''}\mathrm{Det} \qquad M_{\bar{\xi}}[\phi] \neq M[\phi]$$

Example – -$\lambda\phi^{\mathbf{4}}$

Example – $-\lambda\phi^4$

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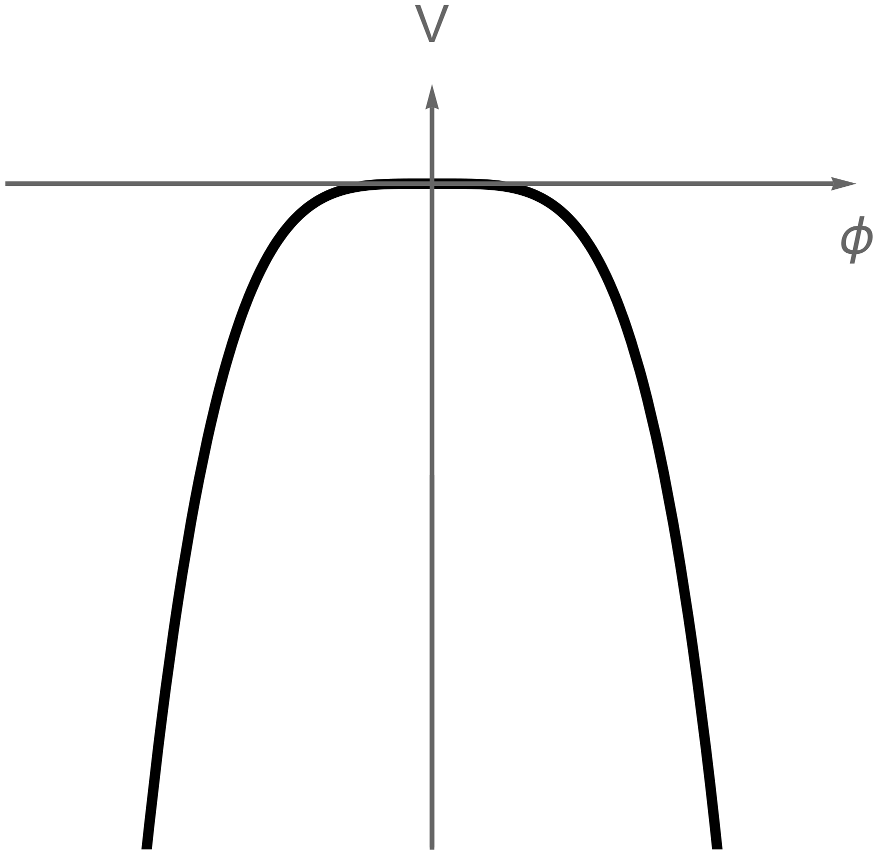

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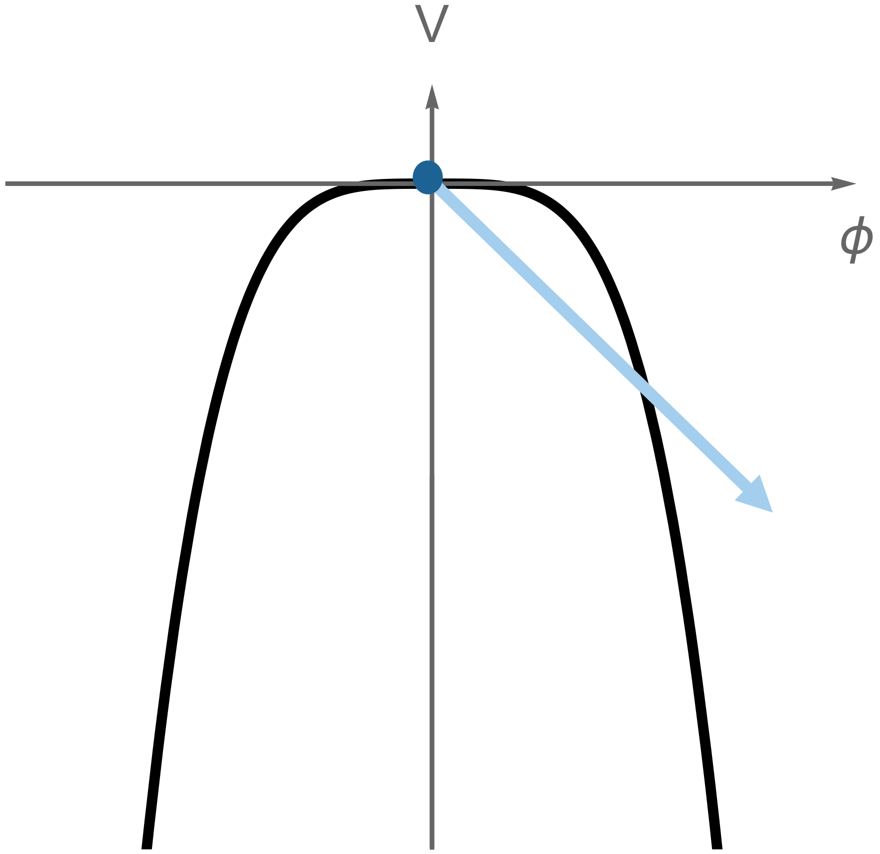

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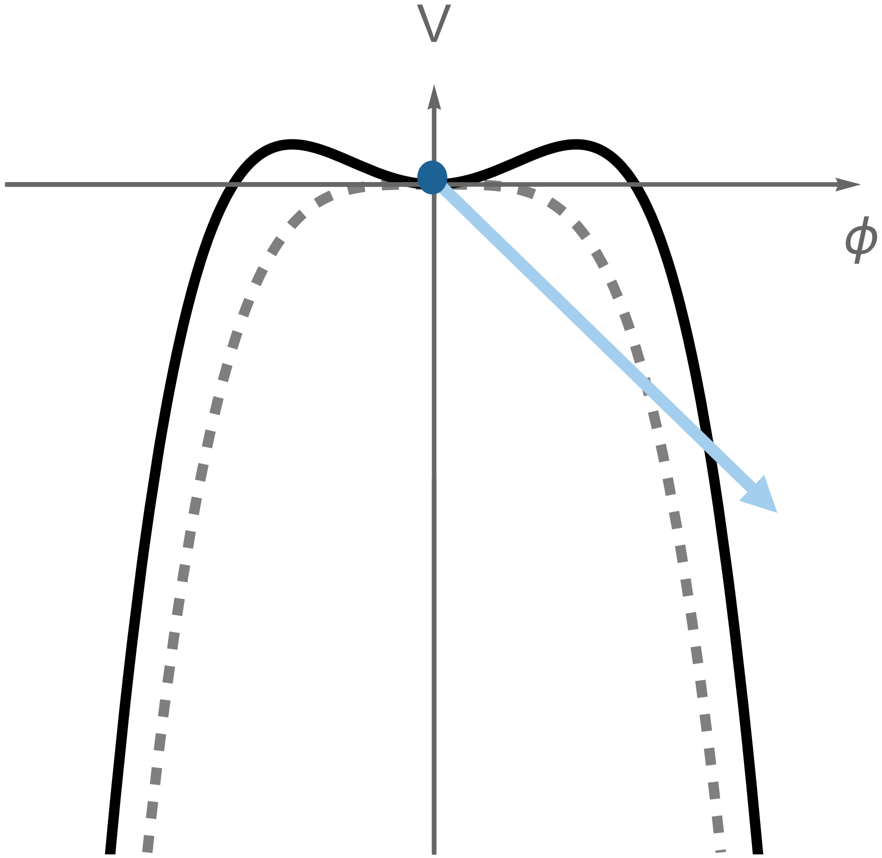

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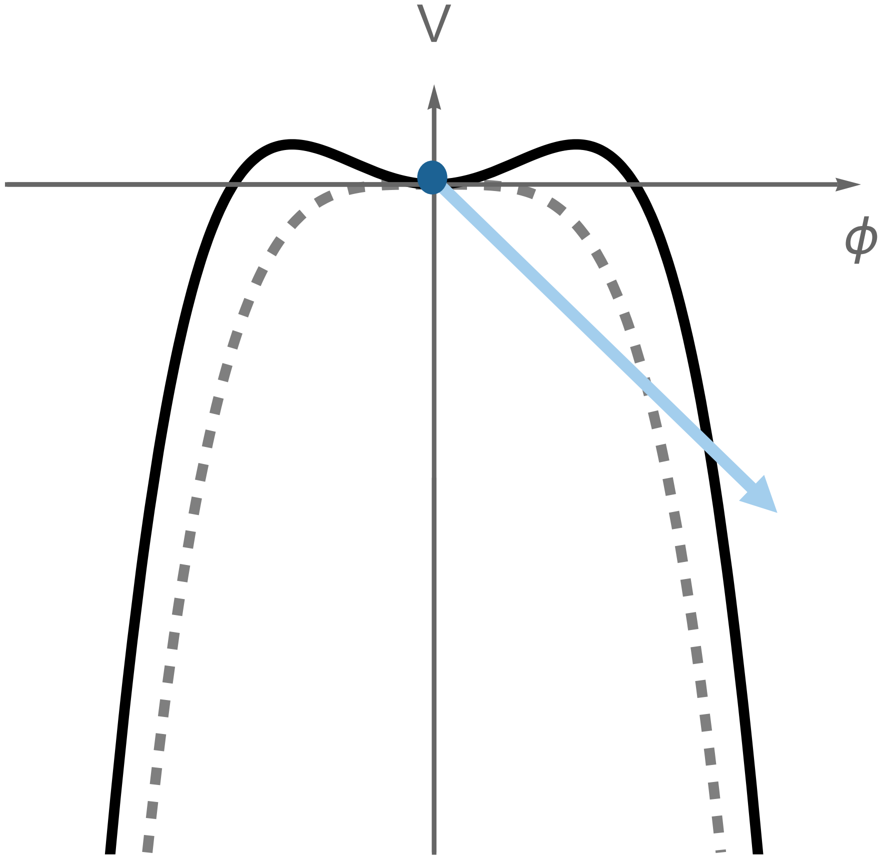

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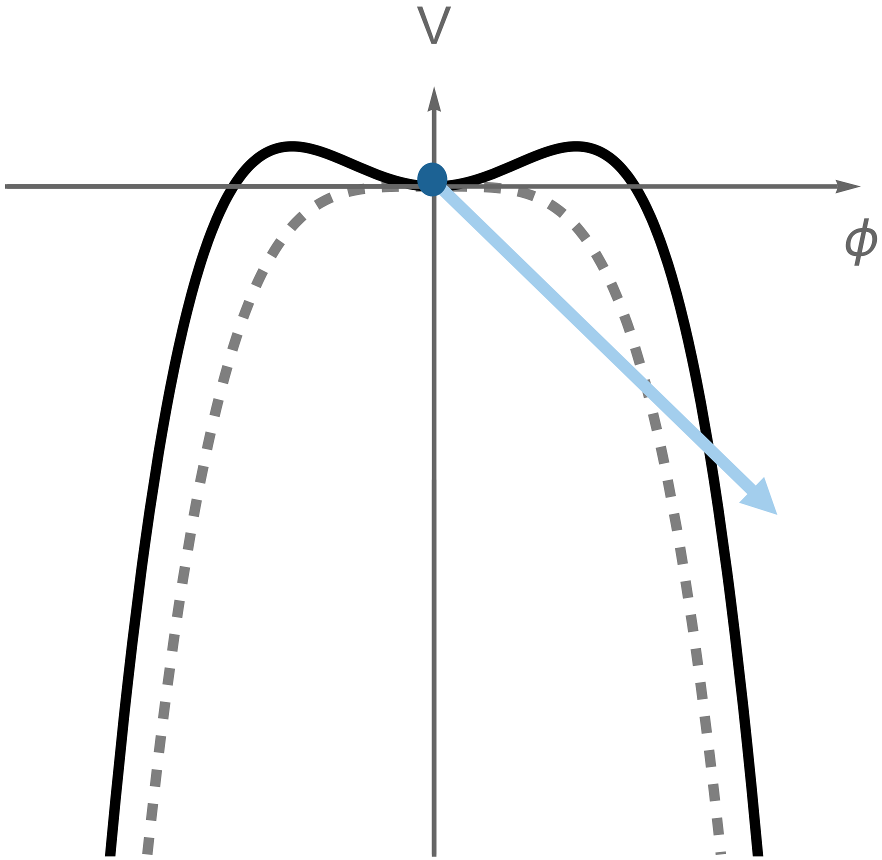

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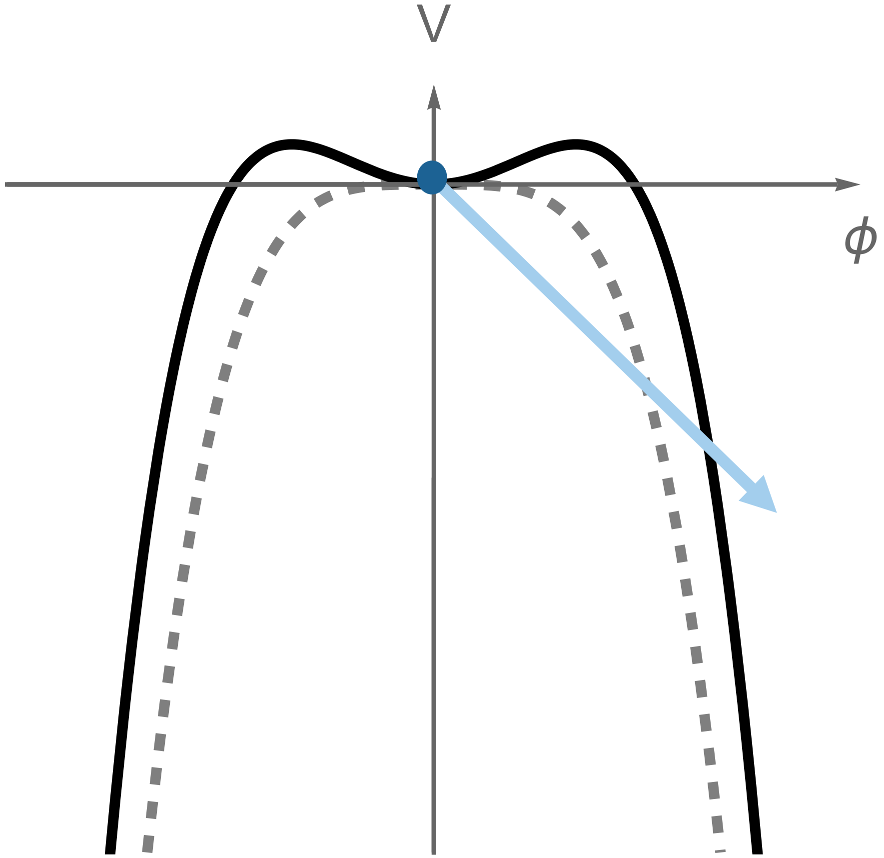

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CULPRIT

$$\phi(x) \to a\phi(ax)$$

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Theory has $O(4)$ symmetry → assume the solution has the same symmetry

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- **Choice of the constraint operator $\mathcal{O}(\phi)$: two different constraints - ϕ^3 and ϕ^6**

Results - action

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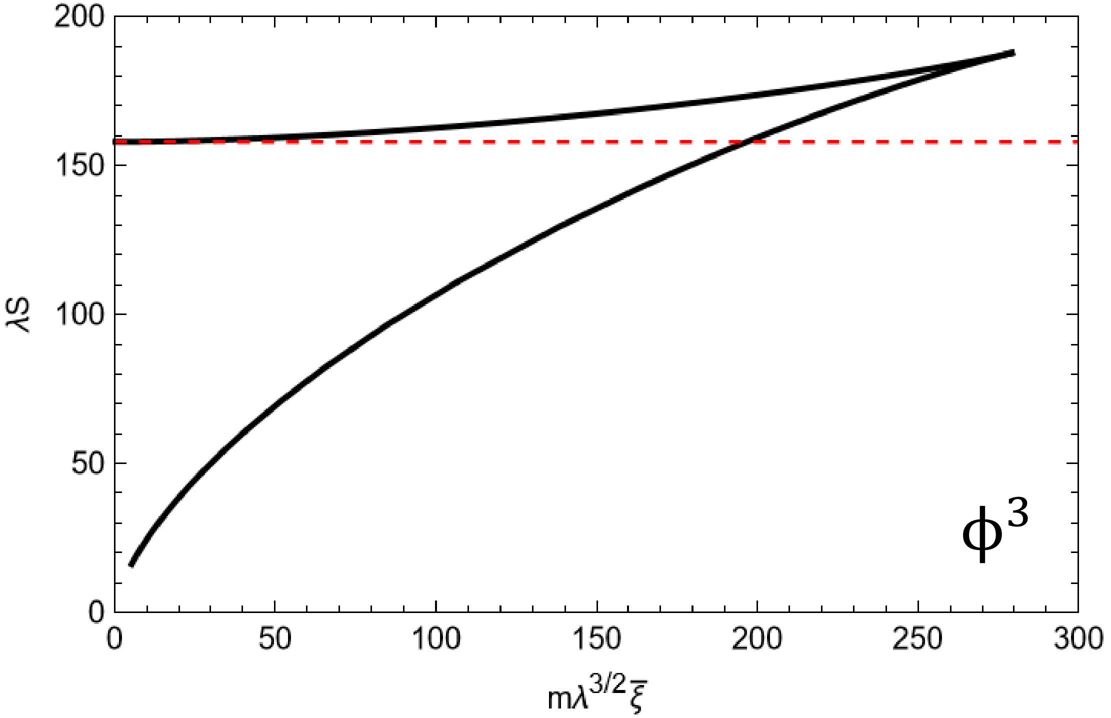

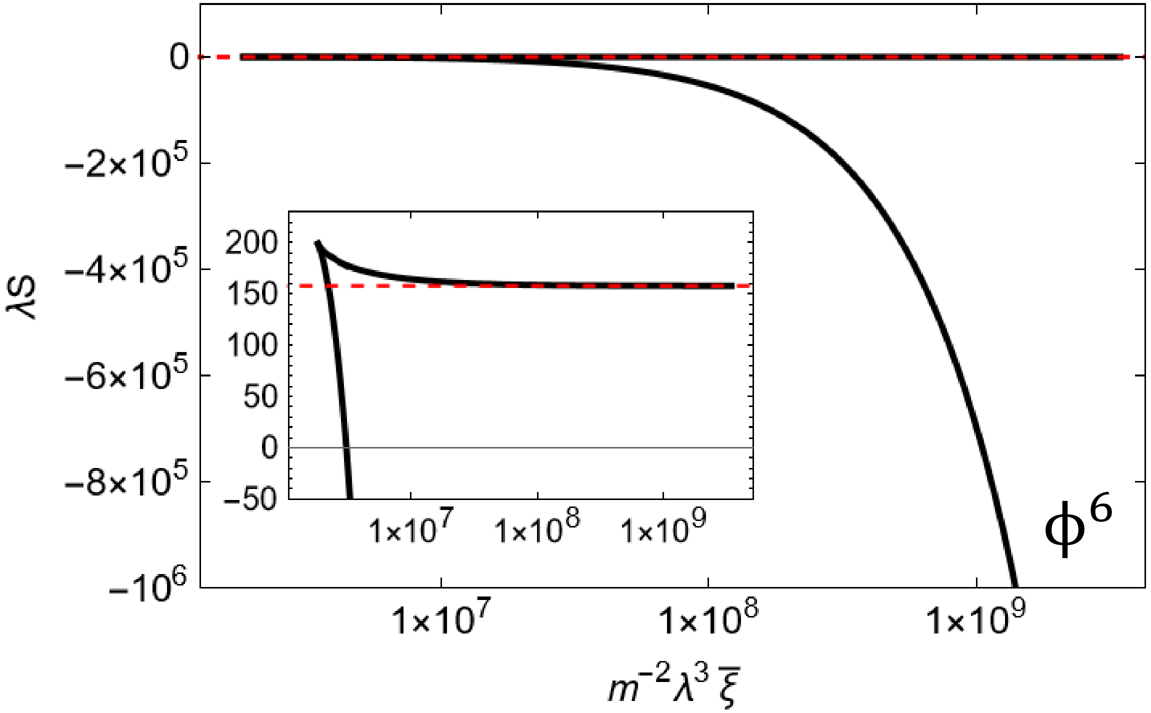

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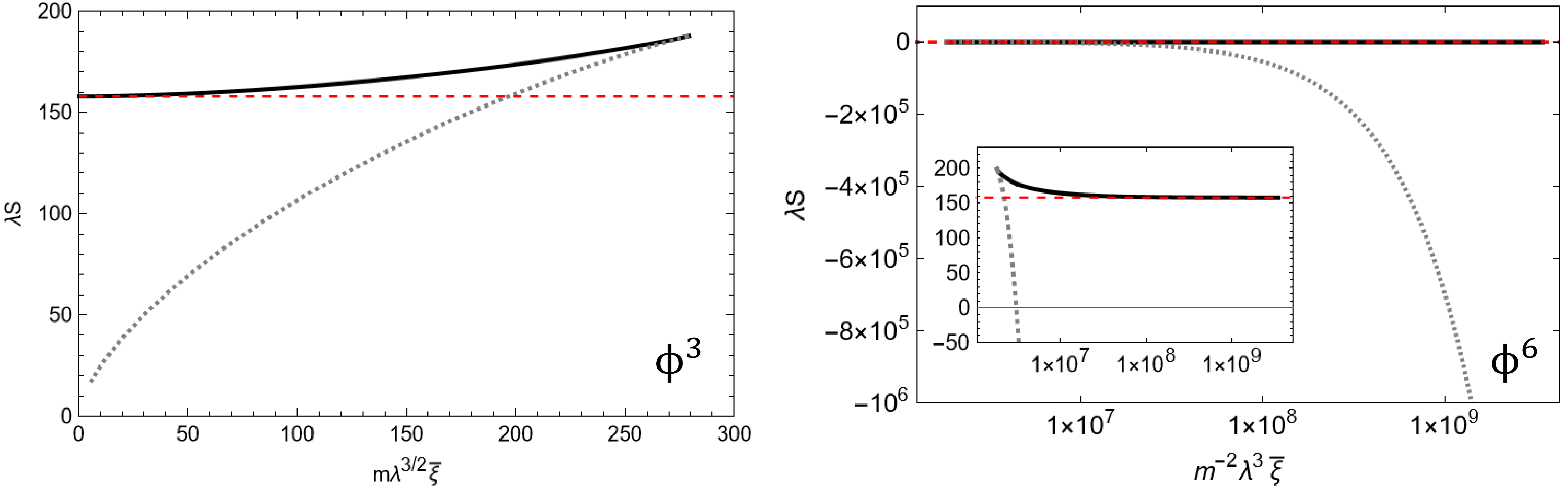

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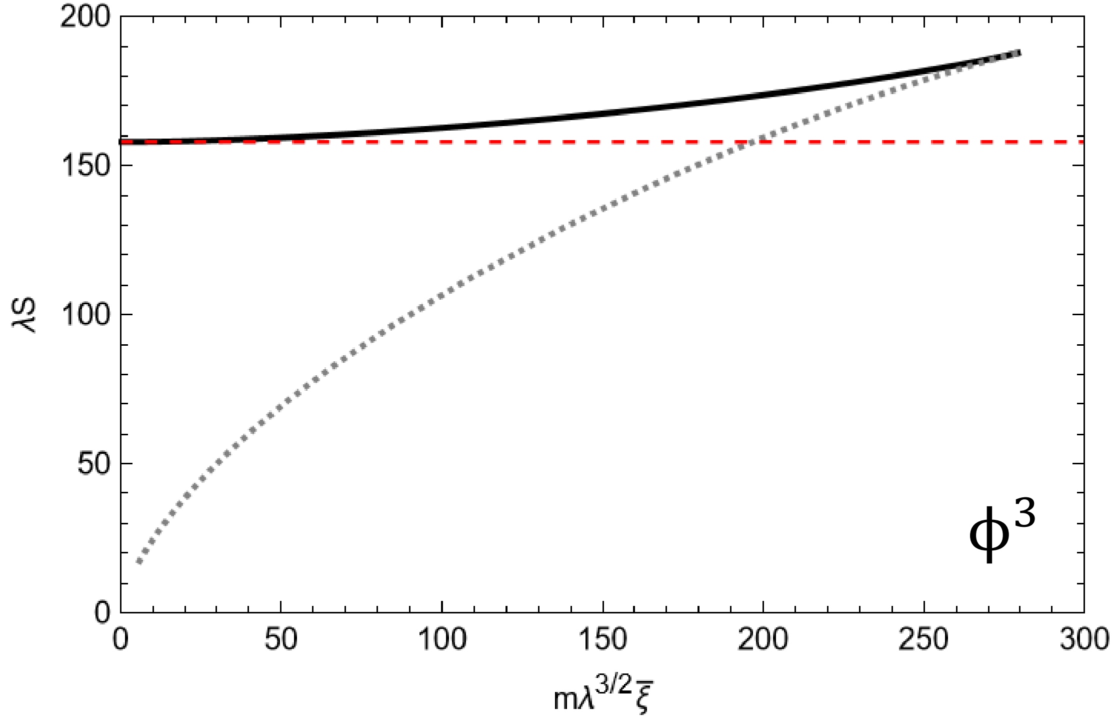

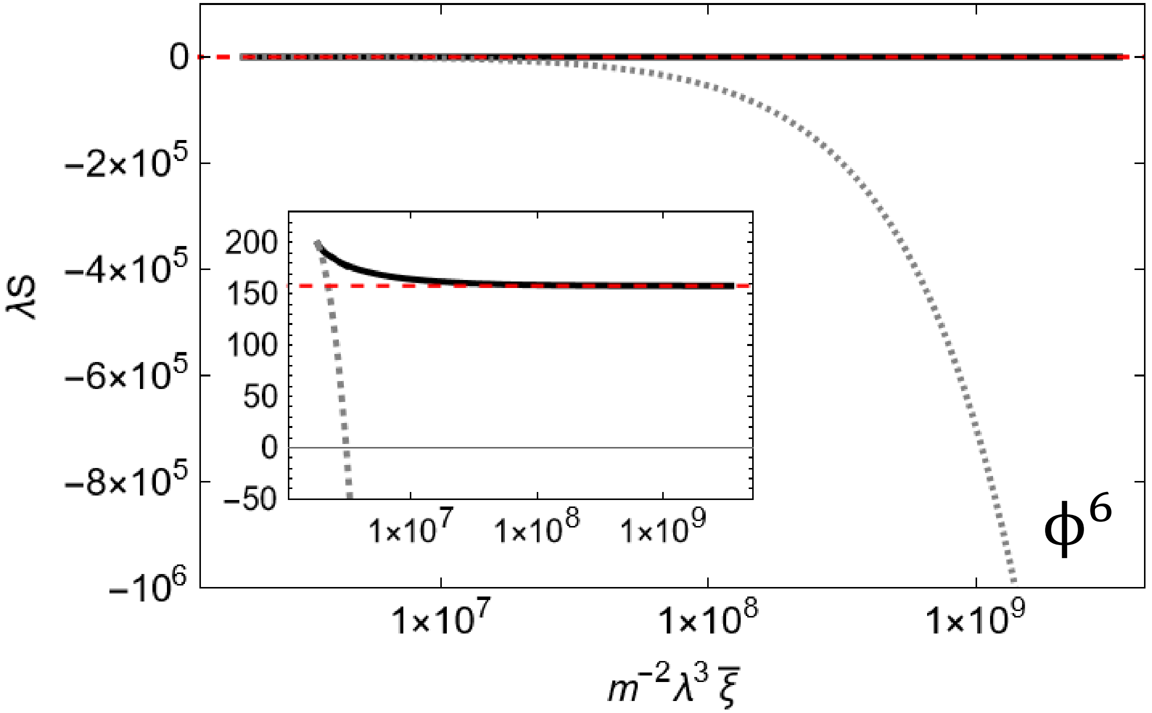

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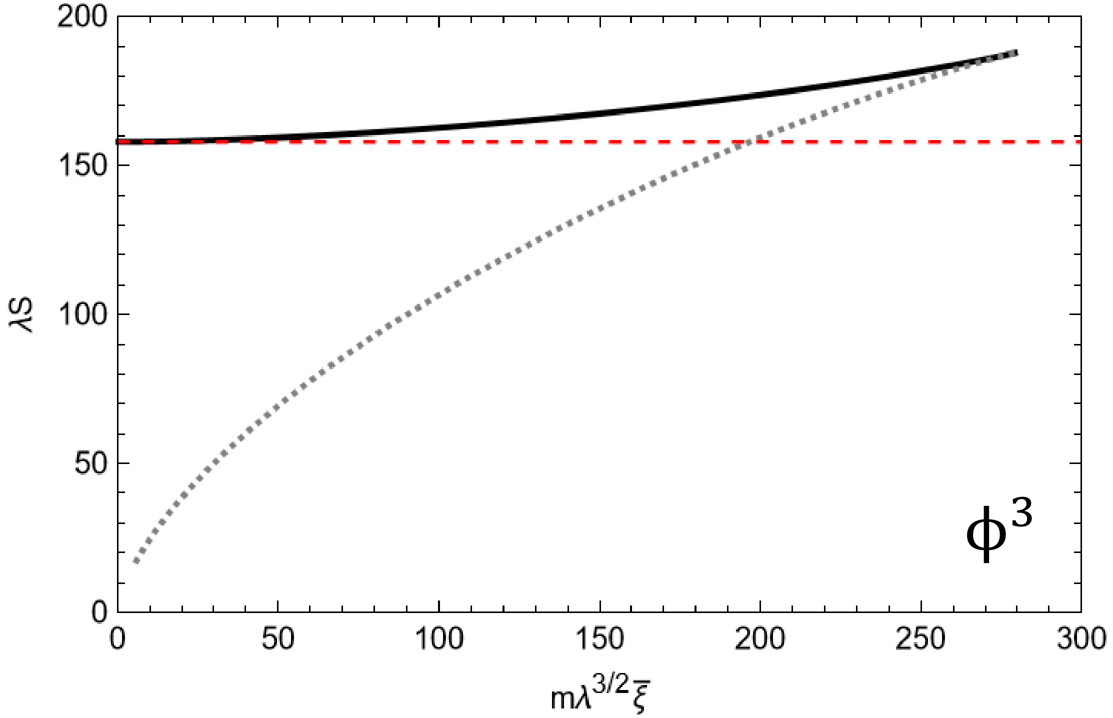

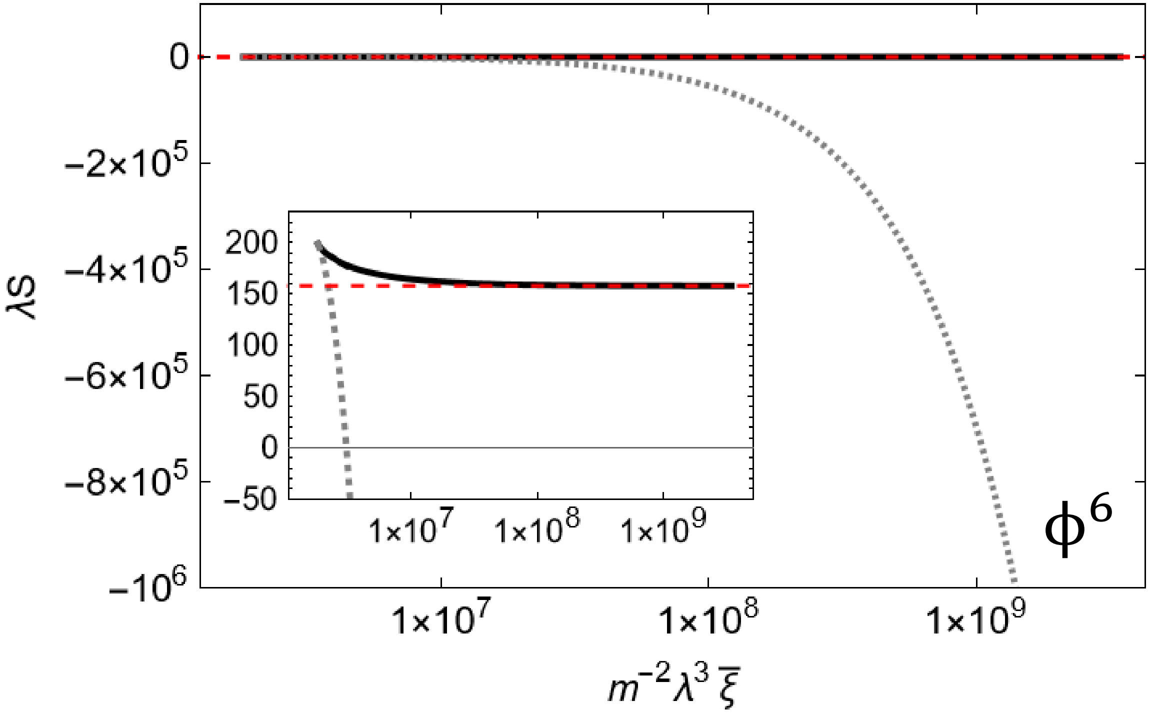

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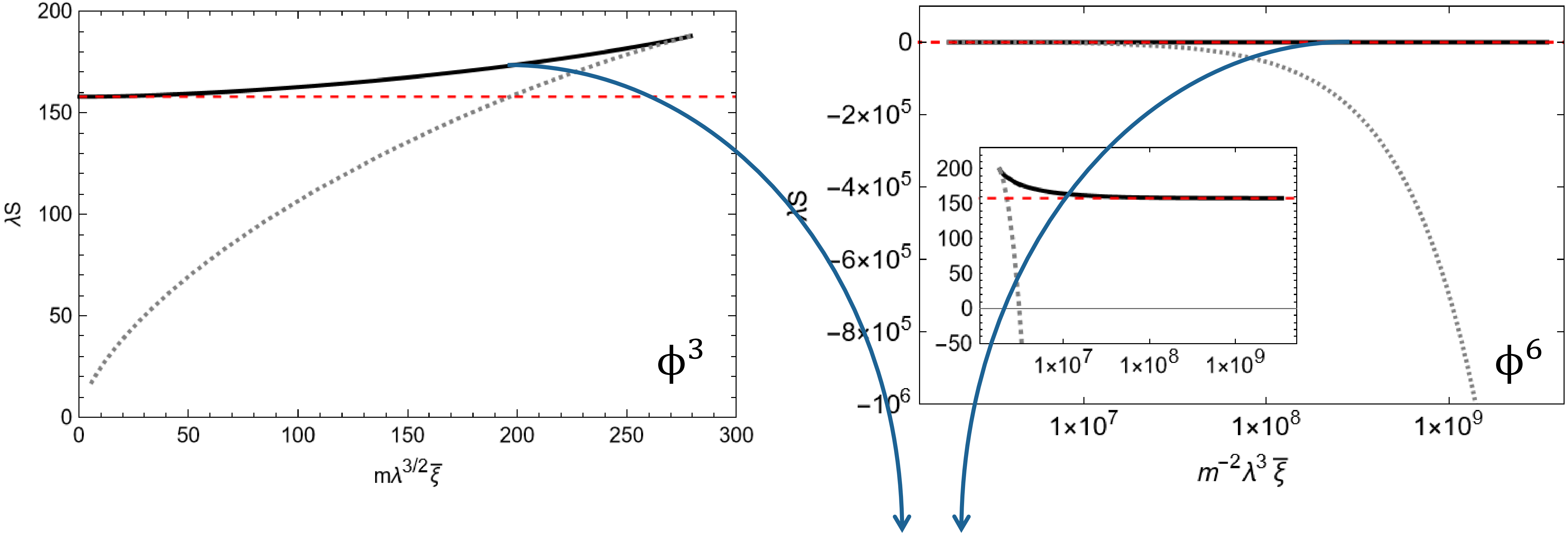

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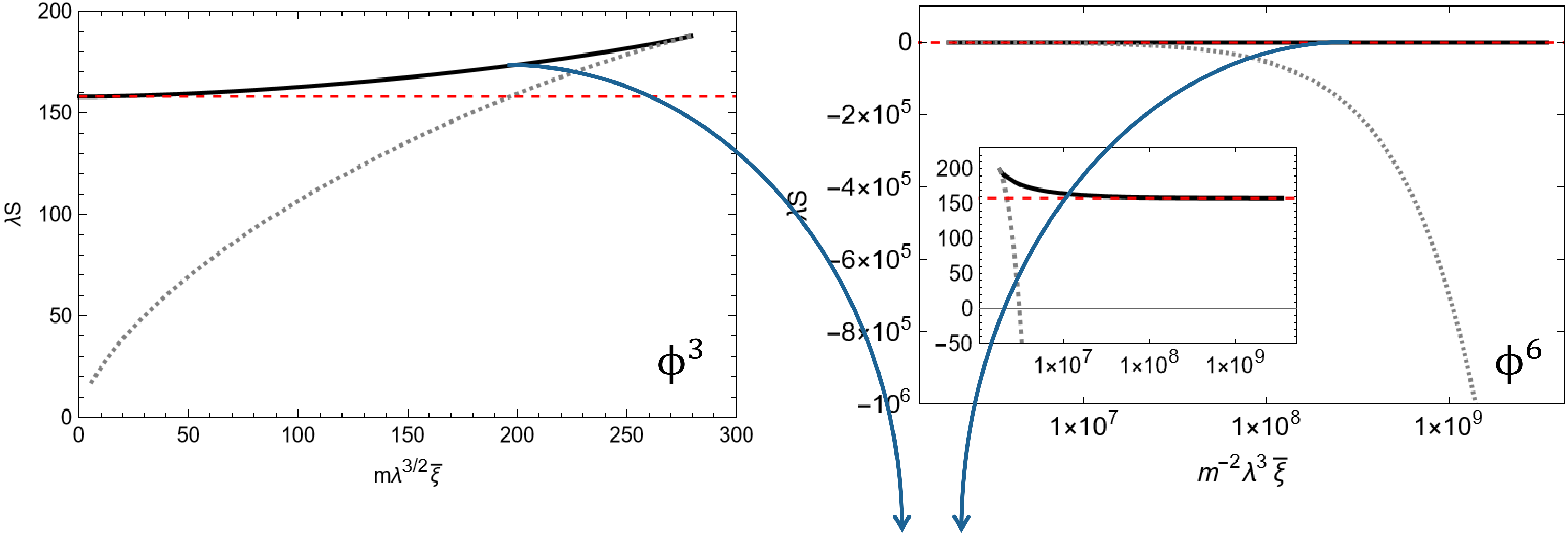

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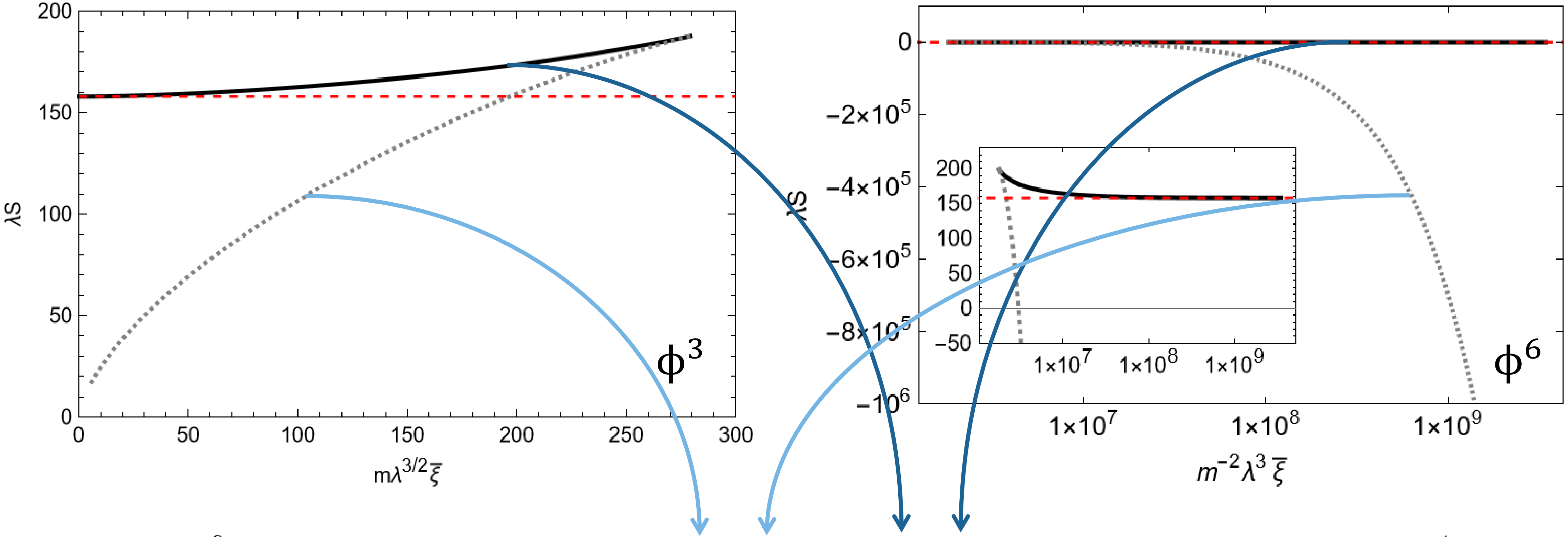

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Results – field profiles

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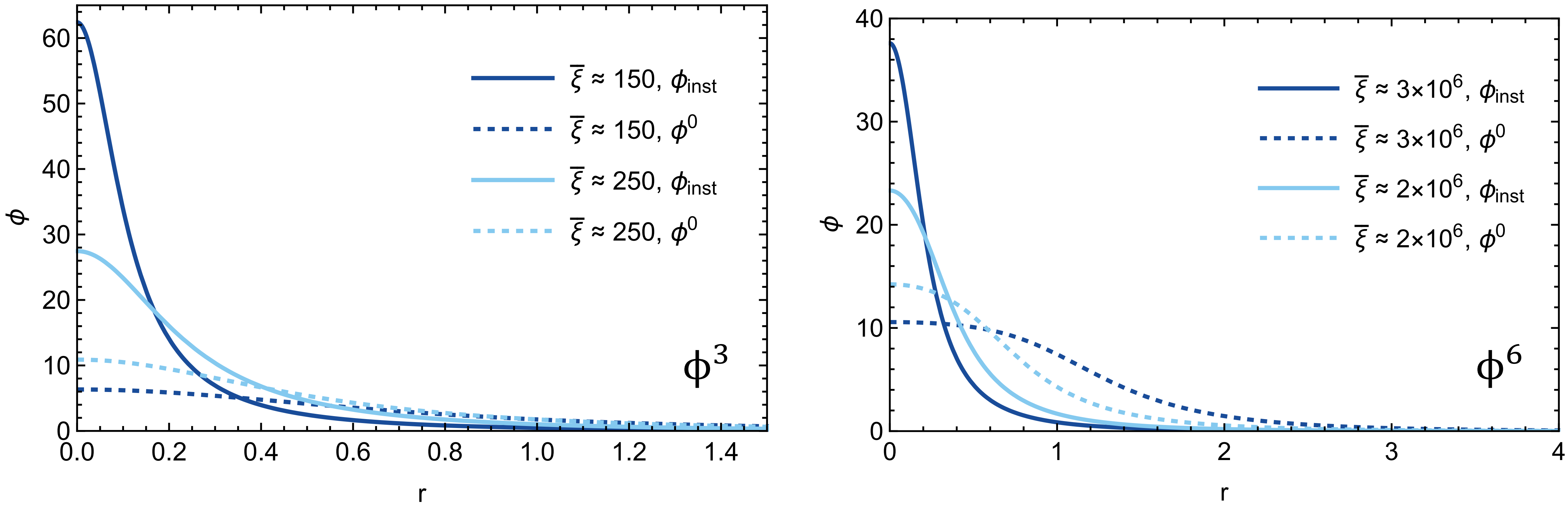

Summary and further research

SUMMARY

- **Introduced a non-perturbative constrained instanton method** for computing vacuum decay rates in theories with no explicit instanton solution
- **Showed you an application** of this method to a particular scalar field theory
- **Omitted a lot of technical details** – these can be found in the paper (out soon)

or just ask me!

FURTHER RESEARCH

- Computation of the **functional determinant** (work in progress)
- **Comparison of the results** (full rate) for different constraint operators
- **Analytic construction of constrained instantons** for extreme values of $\bar{\xi}$ (work in progress)
- **Extension of the method to theories with gauge fields** (partial results) and **application to the electroweak sector of the Standard Model**

Thank you

Backup slides

Lagrange multipliers

In practice? → Lagrange multipliers

Saddle point of $S_E[\phi]$ for some $\xi[\phi] = \bar{\xi}$ $=$ **Saddle point $\tilde{\phi}_\kappa$ of $\tilde{S}_\kappa[\phi]$ for $\xi[\tilde{\phi}_\kappa] = \bar{\xi}$**

MODIFIED ACTION

$$\tilde{S}_\kappa[\phi] = S_E[\phi] + \kappa\xi[\phi]$$

Constant Lagrange multiplier

- Note – different than the standard textbook approach to Lagrange multipliers
- Look for solutions for fixed κ (easy) instead of fixed $\bar{\xi}$ (hard) → Solve EOM following from $\tilde{S}_\kappa$

Equation of motion

$$\tilde{S}_\kappa[\Phi] = \frac{1}{\lambda}\int d^4X \left(\frac{1}{2}(\partial_\mu\Phi)^2 + \frac{1}{2}\Phi^2 - \frac{1}{4!}\Phi^4 + \kappa_n\Phi^n\right)$$

CHOICE

$$n = 3, 6$$

+ $O(4)$ symmetry

$$\Phi'' + \frac{3}{R}\Phi' - \Phi + \frac{1}{6}\Phi^3 + n\kappa_n\Phi^{n-1} = 0$$

Boundary conditions: $\Phi'(R_{\min}) = 0$ and $\Phi(R_{\max}) = 0$

Finite simulation box size

$$R \equiv mr$$

$$\Phi \equiv \frac{\lambda^{1/2}}{m}\phi$$

$$\kappa_n = \frac{m^{n-4}}{\lambda^{n/2-1}}\kappa$$

Results - constraints