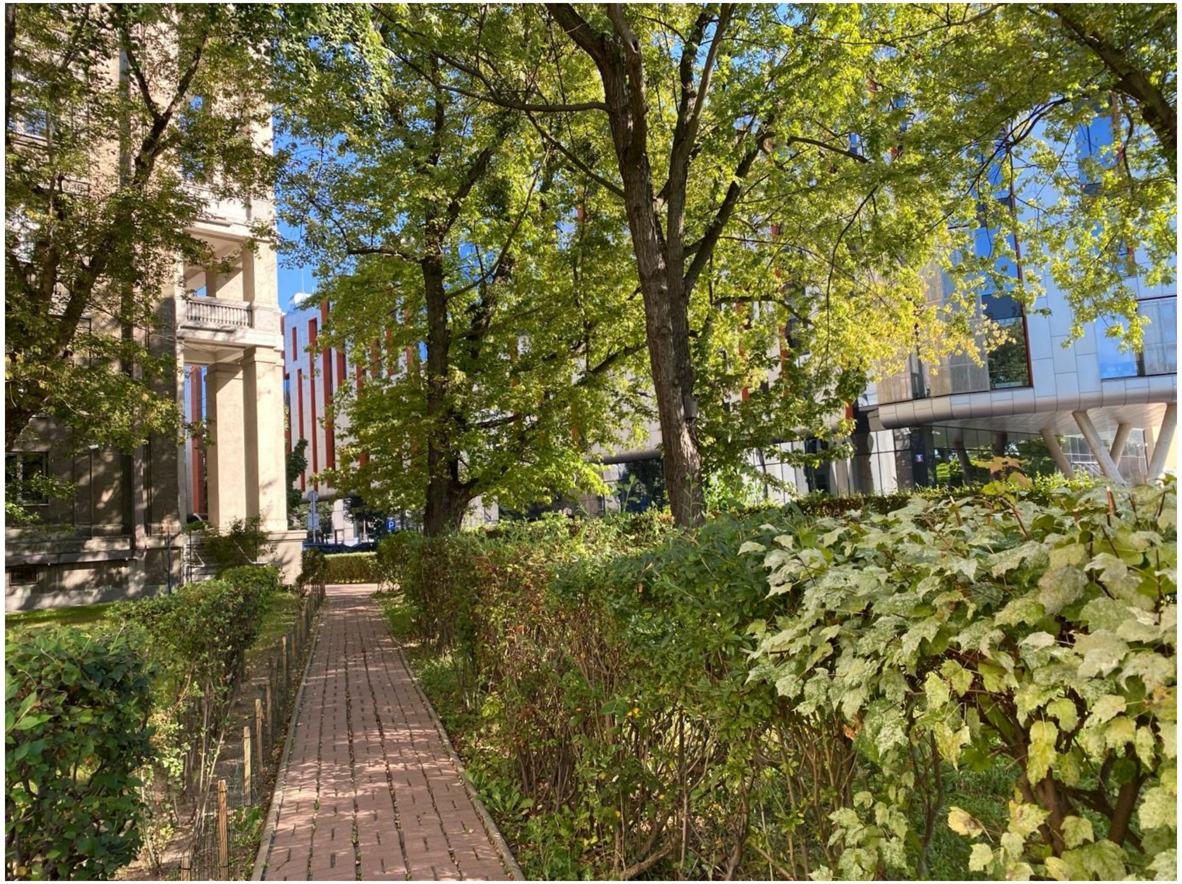
RG-stable parameter relations in absence of a conventional symmetry



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Reference: H.E. Haber and P.M. Ferreira, Eur. Phys. J. C **85**, 541 (2025); Erratum-ibid. C **85**, 867 (2025) [arXiv:2502.11011].

Introduction–a folk theorem

In quantum field theory, generic tree-level parameter relations are not stable under renormalization group (RG) running.

If a tree-level parameter relation is the result of an unbroken symmetry, then the corresponding relation is RG-stable.

In the case of a softly-broken symmetry, the tree-level relations satisfied by dimensionless couplings are RG stable, although they receive *finite* radiative corrections.

Is the converse of the above statements true?

Folk theorem: the presence of an RG-stable parameter relation implies the existence of a symmetry.

A recent result, obtained by Ferreira, Grzadkowski, Ogreid, and Osland in the context of two-Higgs doublet model (2HDM), appears to violate this folk theorem as applied in the context of a conventional symmetry.[1] These authors attempted to resurrect the folk theorem by proposing a rather unconventional symmetry.

Subsequently, Trautner revisited the folk theorem and asserted that it could be validated by expanding the class of allowed symmetry transformations.[2]

Can the observed RG-stable parameter relations be understood within the context of conventional symmetries alone?

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[1] P.M. Ferreira, B. Grzadkowski, O.M. Ogreid and P. Osland, Eur. Phys. J. C **84** (2024) 234, arXiv:2306.02410 [hep-ph].

[2] A. Trautner, arXiv:2505.00099 [to be published in JHEP].

An RG-stable parameter relation without a symmetry

Consider the following 2HDM scalar potential parameter relations, $m_{11}^2 = -m_{22}^2$; $\lambda_1 = \lambda_2$; $\lambda_6 = -\lambda_7$. Then,[3]

$$\beta_{m_{11}^2+m_{22}^2}\Big|_{\rm sym} \equiv \left[\beta_{m_{11}^2} + \beta_{m_{22}^2}\right]\Big|_{\rm sym} = 0\,,$$

$$\beta_{\lambda_1-\lambda_2}\Big|_{\rm sym} \equiv \left[\beta_{\lambda_1} - \beta_{\lambda_2}\right]\Big|_{\rm sym} = 0\,,$$

$$\beta_{\lambda_6+\lambda_7}\Big|_{\rm sym} \equiv \left[\beta_{\lambda_6} + \beta_{\lambda_7}\right]\Big|_{\rm sym} = 0\,.$$

The parameter relation, $m_{11}^2 = -m_{22}^2$ appears to be RG-stable to all orders in perturbation theory.[4] Yet, there is no conventional symmetry that imposes such a relation.[5]

[3]The notation "sym" indicates that the parameter relations were used when evaluating the beta functions.

[4]As shown by P.M. Ferreira et al., the RG-stability holds to all orders in perturbation theory in the quartic scalar couplings and the gauge couplings. It has also been shown to hold at two-loop order in the Yukawa couplings (but no corresponding all-orders result has yet been obtained).

[5]In contrast, the RG-stability of the parameter relations $\lambda_1 = \lambda_2$; $\lambda_6 = -\lambda_7$ is due to a GCP2 symmetry ($\Phi_1 \to \Phi_2^*$ and $\Phi_2 \to -\Phi_1^*$), which is softly broken by the squared-mass terms of the scalar potential.

GOOFy symmetry?

Reference: B. Grzadkowski, O.M. Ogreid, P. Osland, and P.M. Ferreira, op. cit.

Consider the following scalar field transformations:

$$\Phi_1 \to -\Phi_2^* \,, \quad \Phi_1^* \to \Phi_2 \,, \quad \Phi_2 \to \Phi_1^* \,, \quad \Phi_2^* \to -\Phi_1 \,.$$

This is unconventional since Φ_1^* does not transform into the complex conjugate of the transformed Φ_1 (and similarly for Φ_2). Equivalently,

$$\Phi_1^\dagger\Phi_1 \to -\Phi_2^\dagger\Phi_2 \,, \qquad \Phi_2^\dagger\Phi_2 \to -\Phi_1^\dagger\Phi_1 ,$$

whereas $\Phi_1^\dagger\Phi_2$ and $\Phi_2^\dagger\Phi_1$ are invariant. Imposing this "GOOFy symmetry" on the scalar potential yields the parameter relations, $m_{11}^2 = -m_{22}^2$; $\lambda_1 = \lambda_2$; $\lambda_6 = -\lambda_7$.

Moreover, the kinetic energy terms, $\mathscr{L}_{\rm KE} \equiv \sum_i D_\mu \Phi_i^\dagger D^\mu \Phi_i$, change sign under $\Phi_1^\dagger \Phi_1 \to -\Phi_2^\dagger \Phi_2$ and $\Phi_2^\dagger \Phi_2 \to -\Phi_1^\dagger \Phi_1$.

In order to restore the sign of $\mathscr{L}_{\rm KE}$, the authors of the GOOFy paper advanced the radical proposal where the spacetime coordinates themselves also transform under the GOOFy symmetry via $x_\mu \to i x_\mu$.[6]

In particular, there appears to be no conventional symmetry explanation for the RG-stable parameter relation $m_{11}^2 = -m_{22}^2$.

[6] Equivalently, the covariant derivative must also transform as $D_\mu \to i D_\mu$ (which implies that the gauge fields themselves must also similarly transform) in order that the kinetic energy terms of the scalar fields remain invariant.

A toy model with one complex scalar field

Consider a theory of one complex scalar field Φ with Lagrangian,

$$\mathscr{L} = \partial_\mu \Phi \partial^\mu \Phi^* - m_1^2 \Phi^* \Phi - m_2^2 \Phi^2 - m_2^{2*} \Phi^{*2} - \lambda_1 (\Phi^* \Phi)^2$$
$$-\lambda_2 \Phi^4 - \lambda_2^* \Phi^{*4} - \left(\lambda_3 \Phi^2 + \lambda_3^* \Phi^{*2}\right) \Phi^* \Phi \,,$$

after imposing a discrete symmetry $\Phi \to -\Phi$ to remove terms linear and cubic in the scalar fields. Next, impose the relations:

$$m_1^2 = \lambda_3 = 0 \,.$$

These parameter relations are RG-stable to all orders in perturbation theory:

$$\beta_{m_1^2}\Big|_{\rm sym} = 0 \,, \qquad \beta_{\lambda_3}\Big|_{\rm sym} = 0 \,,$$

where "sym" means that the β's are evaluated at $m_1^2 = \lambda_3 = 0$.

The symmetry transformation $\Phi \to i\Phi$ would set $m_2^2 = \lambda_3 = 0$. This symmetry is softly broken, which explains why the relation $\lambda_3 = 0$ is RG-stable. But why is $m_1^2 = 0$ RG-stable?

A GOOFy-like symmetry?

The "symmetry" transformation, $\Phi \to \Phi\,;\ \Phi^* \to -\Phi^*$, removes the terms $m_1^2\Phi^*\Phi + \left(\lambda_3\Phi^2 + \lambda_3^*\Phi^{*2}\right)\Phi^*\Phi$. That is, $m_1^2 = \lambda_3 = 0$.

Once again, the complex conjugate of the Φ transformation is not equal to the Φ^* transformation. Moreover, one must restore the sign of the kinetic energy term $\mathscr{L}_{\rm KE} = \partial_\mu\Phi\partial^\mu\Phi^*$ by transforming $x_\mu \to ix_\mu$. That is, there seems to be no conventional symmetry explanation for the RG-stability of $m_1^2 = 0$.

Realification of a complex scalar field theory

One can always "realify" a complex scalar field theory by writing $\Phi = (\varphi_1 + i\varphi_2)/\sqrt{2}$. After imposing the GOOFy-like symmetry,

$$\mathcal{V} = \tfrac{1}{2}m_{11}^2\,\left(\varphi_1^2 - \varphi_2^2\right) + m_{12}^2\,\varphi_1\varphi_2 + \tfrac{1}{24}\lambda_{1111}\,\left(\varphi_1^4 + \varphi_2^4\right)$$
$$+\tfrac{1}{4}\lambda_{1122}\,(\varphi_1\varphi_2)^2 + \tfrac{1}{6}\lambda_{1112}\,\left(\varphi_1^2 - \varphi_2^2\right)\,\varphi_1\varphi_2\,.$$

In particular, there are three parameter relations:[7]

$$m_{22}^2 = -m_{11}^2 \qquad \lambda_{1111} = \lambda_{2222}\,, \qquad \lambda_{1112} = -\lambda_{1222}\,,$$

which are equivalent to the previous relations, $m_1^2 = \lambda_3 = 0$.

[7] The notation corresponds to $\mathcal{V} = \frac{1}{2}m_{ij}^2\varphi_i\varphi_j + \frac{1}{24}\lambda_{ijk\ell}\varphi_i\varphi_j\varphi_k\varphi_\ell$, with an implicit sum over repeated indices. The specific model above was first proposed in a talk by B. Grzadkowski at the at the 2024 Workshop on Multi-Higgs Models in Lisbon, Portugal.

The symmetry transformations $\varphi_1 \to \varphi_2$; $\varphi_2 \to -\varphi_1$ would set $m_{11}^2 = m_{22}^2$; $m_{12}^2 = 0$; $\lambda_{1111} = \lambda_{2222}$; $\lambda_{1112} = -\lambda_{1222}$. This symmetry is softly broken, so that the quartic coupling relations are RG-stable. But why is $m_{11}^2 = -m_{22}^2$ RG-stable?

The corresponding GOOFy-like "symmetry" transformations that yield $m_{11}^2 = -m_{22}^2; \lambda_{1111} = \lambda_{2222}$; $\lambda_{1112} = -\lambda_{1222}$ are:[8]

$$\varphi_1 \to i\varphi_2 \,, \qquad\qquad \varphi_2 \to -i\varphi_1 \,.$$

This is not a conventional symmetry of a *real* scalar field theory.[9] Nevertheless, the parameter relation $m_{22}^2 = -m_{11}^2$ is RG-stable to all orders of perturbation theory. Indeed,

$$\beta_{m_{11}^2+m_{22}^2}\Big|_{\rm sym} = \beta_{m_{11}^2} + \beta_{m_{22}^2}\Big|_{\rm sym} = 0 \,,$$

[8]These relations are similar to the 2HDM parameter relations imposed by the GOOFy symmetry.

[9]Moreover, this "symmetry" transformation flips the sign of the kinetic energy terms.

Explaining the mysterious RG-stability: Complexification

The unconventional symmetry

$$\varphi_1 \to i\varphi_2\,, \qquad\qquad \varphi_2 \to -i\varphi_1\,.$$

would have been a conventional symmetry if the φ_i had been complex scalar fields.

Our proposal is to promote the two real fields φ_1 and φ_2 to complex fields Φ_1 and Φ_2, and then impose the conventional symmetry transformations

$$\Phi_1 \to i\Phi_2\,, \qquad\qquad \Phi_2 \to -i\Phi_1\,.$$

We will additionally impose a CP symmetry, $\Phi_i \to \Phi_i^*$, to impose reality conditions on the parameters of the scalar potential.

The resulting parameter relations of the complexified theory are RG-stable due to these symmetries.

Moreover, we will show that the corresponding β-function relations of the complexified theory can be related to β-function relations obtained in the original real scalar field theory.

That is, the RG-stability of the parameter relations of the original theory can be attributed to conventional symmetries of the complexified theory, thereby restoring the original folk theorem.

The complexification recipe

- Promote the real scalar fields φ_i to complex fields Φ_i.
- The complexified model is *defined* to employ a canonically normalized kinetic energy term,

$$\mathscr{L}_{\rm KE} = \partial^\mu \Phi^*_{\bar{a}} \partial_\mu \Phi_a \,,$$

- Impose the appropriate $\mathbb{Z}_2$ symmetry to eliminate terms with an odd number of fields.
- Promote the GOOFy-like symmetries of the real scalar field theory to conventional symmetries of the complexified theory.
- Impose a CP symmetry so that the scalar potential parameters are real.

Complex index notation

The kinetic energy term, $\mathscr{L}_{\rm KE} = \partial^\mu \Phi^*_{\bar{a}} \partial_\mu \Phi_a$, is invariant under a U(2) basis transformation,[10]

$$\Phi_a \to U_{a\bar{b}} \Phi_b \,, \qquad \Phi^*_{\bar{a}} \to \Phi^*_{\bar{b}} U^\dagger_{b\bar{a}} \,,$$

where $U^\dagger_{b\bar{a}} U_{a\bar{c}} = \delta_{b\bar{c}}$. In the notation introduced above, the indices $a, b, c \in \{1, 2\}$ and $\bar{a}, \bar{b}, \bar{c} \in \{\bar{1}, \bar{2}\}$ run over the complex two dimensional flavor space of the scalar fields. The use of the unbarred/barred index notation is accompanied by the rule that there is an implicit sum over unbarred/barred index pairs.

[10] In a different (and perhaps more common) index convention, unbarred indices are lower indices and barred indices are upper indices, with the rule that one sums over upper/lower index pairs. For notational reasons, we preferred the unbarred/barred index notation in this work.

Scalar potential of the complexified model

After removing terms with an odd number of fields, the most general renormalizable scalar potential is:

$$\mathcal{V}_C = M^2_{a\bar{b}}\Phi^*_{\bar{a}}\Phi_b + M^2_{\bar{a}\bar{b}}\Phi_a\Phi_b + M^2_{ab}\Phi^*_{\bar{a}}\Phi^*_{\bar{b}} + \Lambda_{ab\bar{c}\bar{d}}\,\Phi^*_{\bar{a}}\Phi^*_{\bar{b}}\Phi_c\Phi_d$$
$$+\Lambda_{\bar{a}\bar{b}\bar{c}\bar{d}}\,\Phi_a\Phi_b\Phi_c\Phi_d + \Lambda_{a\bar{b}\bar{c}\bar{d}}\,\Phi^*_{\bar{a}}\Phi_b\Phi_c\Phi_d + \Lambda_{abc\bar{d}}\,\Phi^*_{\bar{a}}\Phi^*_{\bar{b}}\Phi^*_{\bar{c}}\Phi_d + \Lambda_{abcd}\Phi^*_{\bar{a}}\Phi^*_{\bar{b}}\Phi^*_{\bar{c}}\Phi^*_{\bar{d}}\,.$$

In this notation, M^2_{ab} and $M^2_{a\bar{b}}$ are independent tensors (despite the use of the same symbol M^2). Likewise, Λ_{abcd}, $\Lambda_{abc\bar{d}}$, and $\Lambda_{ab\bar{c}\bar{d}}$ are independent tensors (despite the use of the same symbol Λ).

Hermiticity and permutation symmetry imply

$$M^2_{\bar{a}\bar{b}} = M^2_{\bar{b}\bar{a}}\,, \qquad M^2_{ab} = M^2_{ba}\,, \qquad \Lambda_{ab\bar{c}\bar{d}} = \Lambda_{ba\bar{c}\bar{d}} = \Lambda_{ab\bar{d}\bar{c}} = \Lambda_{ba\bar{d}\bar{c}}\,,$$
$$M^2_{a\bar{b}} = \left[M^2_{b\bar{a}}\right]^*, \qquad \Lambda_{ab\bar{c}\bar{d}} = \left[\Lambda_{cd\bar{a}\bar{b}}\right]^*,$$
$$M^2_{\bar{a}\bar{b}} = \left[M^2_{ab}\right]^*, \qquad \Lambda_{\bar{a}\bar{b}\bar{c}\bar{d}} = \left[\Lambda_{abcd}\right]^*, \qquad \Lambda_{d\bar{a}\bar{b}\bar{c}} = \left[\Lambda_{abc\bar{d}}\right]^*.$$

In particular, $M^2_{1\bar{1}}$, $M^2_{2\bar{2}}$, $\Lambda_{11\bar{1}\bar{1}}$, $\Lambda_{22\bar{2}\bar{2}}$, and $\Lambda_{12\bar{1}\bar{2}} = \Lambda_{21\bar{2}\bar{1}}$ are real parameters.

Imposing the conventional symmetry on the complexified model

If $\mathcal{V}_C$ is invariant under $\Phi_1 \to i\,\Phi_2$ and $\Phi_2 \to -i\,\Phi_1$, then the following parameter relations are obtained:[11]

$$M^2_{1\bar{1}} = M^2_{2\bar{2}} \qquad \mathrm{Re}\, M^2_{1\bar{2}} = 0\,, \qquad M^2_{11} = -M^2_{22}\,,$$

$$\Lambda_{1111} = \Lambda_{2222}\,, \qquad \Lambda_{1112} = -\Lambda_{1222}\,,$$

$$\Lambda_{111\bar{1}} = -\Lambda_{222\bar{2}}\,, \qquad \Lambda_{112\bar{1}} = \Lambda_{122\bar{2}}\,, \qquad \Lambda_{112\bar{2}} = -\Lambda_{122\bar{1}}\,, \qquad \Lambda_{111\bar{2}} = \Lambda_{222\bar{1}}\,,$$

$$\Lambda_{11\bar{1}\bar{1}} = \Lambda_{22\bar{2}\bar{2}}\,, \qquad \Lambda_{11\bar{1}\bar{2}} = -\Lambda^*_{12\bar{2}\bar{2}}\,, \qquad \Lambda_{11\bar{2}\bar{2}} = \Lambda^*_{11\bar{2}\bar{2}}\,.$$

After imposing CP symmetry, which renders all scalar potential parameters real, it follows that $M^2_{1\bar{2}} = 0$, and we are left with 3 real squared-mass terms and 11 real quartic couplings.

[11] The parameter relations in red are the same as those of the original realified toy model.

The resulting scalar potential is depends on three real squared-mass terms and eleven real quartic couplings:

$$M^2_{ab} = M^2_{\bar{a}\bar{b}} \ni \{\bar{M}^2, M^2_{12}\}\,,$$

$$M^2_{a\bar{b}} = M^2_{b\bar{a}} \ni \{M^2\}\,,$$

$$\Lambda_{ab\bar{c}\bar{d}} = \Lambda_{cd\bar{a}\bar{b}} \ni \{\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4\}\,,$$

$$\Lambda_{abcd} = \Lambda_{\bar{a}\bar{b}\bar{c}\bar{d}} \ni \{\Lambda_5, \Lambda_6, \Lambda_7\}\,,$$

$$\Lambda_{abc\bar{d}} = \Lambda_{a\bar{b}\bar{c}\bar{d}} \ni \{\Lambda_8, \Lambda_9, \Lambda_{10}, \Lambda_{11}\}\,,$$

where

$$M^2 \equiv M^2_{1\bar{1}} = M^2_{2\bar{2}}\,, \qquad \bar{M}^2 \equiv M^2_{11} = -M^2_{22}\,,$$

$$\Lambda_1 \equiv \Lambda_{1111} = \Lambda_{2222}\,, \qquad \Lambda_2 \equiv \Lambda_{12\bar{1}\bar{2}}\,, \qquad \Lambda_3 \equiv \Lambda_{11\bar{2}\bar{2}}\,,$$

$$\Lambda_4 \equiv \Lambda_{11\bar{1}\bar{2}} = -\Lambda_{12\bar{2}\bar{2}}\,, \qquad \Lambda_5 \equiv \Lambda_{1122}\,, \qquad \Lambda_6 \equiv \Lambda_{1111} = \Lambda_{2222}\,,$$

$$\Lambda_7 \equiv \Lambda_{1112} = -\Lambda_{1222}\,, \qquad \Lambda_8 \equiv \Lambda_{112\bar{1}} = \Lambda_{122\bar{2}}\,, \qquad \Lambda_9 \equiv \Lambda_{111\bar{1}} = -\Lambda_{222\bar{2}}\,,$$

$$\Lambda_{10} \equiv \Lambda_{112\bar{2}} = -\Lambda_{122\bar{1}}\,, \qquad \Lambda_{11} \equiv \Lambda_{111\bar{2}} = \Lambda_{222\bar{1}}\,.$$

Realification and Complexification summarized

These terms are inspired by their usage in Lie algebra theory.

as applied to scalar field theory	Lie algebra example
theory of one complex scalar Φ	$\mathfrak{sl}(2,\mathbb{C})$
$\downarrow$ realify	$\downarrow$ realify
theory of two real scalars φ_1, φ_2	$\mathfrak{sl}(2,\mathbb{C})_\mathbb{R} \cong \mathfrak{so}(3,1)$
$\downarrow$ complexify	$\downarrow$ complexify
theory of two complex scalars Φ_1, Φ_2	$\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C}) \cong \mathfrak{so}(4,\mathbb{C})$

Note: The Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ is defined to be the set of complex traceless 2×2 matrices. Any element of $\mathfrak{sl}(2,\mathbb{C})$ is given by *complex* linear combination of the three generators, $\left\{ \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \right\}$. The realified version, denoted by $\mathfrak{sl}(2,\mathbb{C})_\mathbb{R}$, consists of *real* linear combinations of the six generators, $\left\{ \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & i \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ i & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right) \right\}$. The realified version is equivalent to the original complex version written in a different way.

β functions of the complexified theory

To use the formulae previously given for a real scalar field theory, we could carry out the realification procedure one more time by defining $\Phi_1 = (\varphi_1 + i\varphi_2)/\sqrt{2}$ and $\Phi_2 = (\varphi_3 + i\varphi_4)/\sqrt{2}$. From the resulting β functions, we have derived the corresponding formulae expressed in terms of the complex parameters of $\mathcal{V}(\Phi_1, \Phi_2)$. At one-loop, we find:

$$\beta_{M^2_{\bar{a}\bar{b}}} = 4M^2_{\bar{c}\bar{d}}\Lambda_{cd\bar{a}\bar{b}} + 24M^2_{cd}\Lambda_{\bar{a}\bar{b}\bar{c}\bar{d}} + 6M^2_{e\bar{d}}\Lambda_{d\bar{a}\bar{b}\bar{e}}\,,$$

$$\beta_{M^2_{a\bar{b}}} = 12M^2_{cd}\Lambda_{a\bar{b}\bar{c}\bar{d}} + 12M^2_{\bar{c}\bar{d}}\Lambda_{acd\bar{b}} + 8M^2_{d\bar{e}}\Lambda_{ae\bar{b}\bar{d}}\,.$$

Apart from the numerical coefficients, the form of these equations is fixed by the index structure of the various terms.

Given that a symmetry of the complexified model imposes the condition $M^2_{11} = -M^2_{22}$, we can write the parameter relation abstractly as

$$c_{ab} M^2_{\bar{a}\bar{b}} = 0\,, \qquad \text{where } c_{11} = c_{22} = 1 \text{ and } c_{12} = c_{21} = 0.$$

The symmetry guarantees that

$$c_{ab}\beta_{M^2_{\bar{a}\bar{b}}}\Big|_{\rm sym} = c_{ab}\left[4M^2_{\bar{c}\bar{d}}\Lambda_{cd\bar{a}\bar{b}} + 24M^2_{cd}\Lambda_{\bar{a}\bar{b}\bar{c}\bar{d}} + 6M^2_{e\bar{d}}\Lambda_{d\bar{a}\bar{b}\bar{e}}\right]\Big|_{\rm sym} = 0\,,$$

where "sym" indicates that the parameter relations of the complexified theory have been applied. But the three quantities in the middle expression above are linearly independent tensors. Thus, each of these quantities must separately vanish.

We conclude that

$$c_{ab} M^2_{\bar{c}\bar{d}} \Lambda_{cd\bar{a}\bar{b}}\Big|_{\rm sym} = 0\,,$$

$$c_{ab} M^2_{cd} \Lambda_{\bar{a}\bar{b}\bar{c}\bar{d}}\Big|_{\rm sym} = 0\,,$$

$$c_{ab} M^2_{e\bar{d}} \Lambda_{d\bar{a}\bar{b}\bar{e}}\Big|_{\rm sym} = 0\,.$$

Compare the result in red font to the vanishing of the one-loop beta function of the original toy model of two real scalar fields:

$$\beta_{c_{ij} m^2_{ij}}\Big|_{\rm sym} = c_{ij} m^2_{k\ell} \lambda_{ijk\ell}\Big|_{\rm sym} = 0\,,$$

with $c_{11} = c_{22} = 1$ and $c_{12} = c_{21} = 0$ and "sym" indicates that the parameter relations of the real theory have been applied.

Since Λ_{abcd} and $\Lambda_{\bar{a}\bar{b}\bar{c}\bar{d}}$ are numerically equal (due to CP symmetry), the two equations in red are algebraically identical.

At two-loop order, $\beta \equiv \beta^I + \beta^{II}$, where

$$\beta^{II}_{m^2_{ij}} = \frac{1}{12}\big(\lambda_{ik\ell m}\lambda_{nk\ell m}m^2_{nj}+\lambda_{jk\ell m}\lambda_{nk\ell m}m^2_{ni}\big)-2m^2_{k\ell}\lambda_{ikmn}\lambda_{j\ell mn}\,.$$

In the original real scalar theory, if the parameter relation $c_{ij}m^2_{ij}$ is RG-stable, then

$$\begin{aligned} c_{ij}\big(\lambda_{ik\ell m}\lambda_{nk\ell m}m^2_{nj} + \lambda_{jk\ell m}\lambda_{nk\ell m}m^2_{ni}\big)\big|_{\rm sym} &= 0\,, \\ c_{ij}m^2_{k\ell}\lambda_{ikmn}\lambda_{j\ell mn}\big|_{\rm sym} &= 0\,. \end{aligned}$$

The corresponding equations for $\beta^{II}_{M^2_{\bar{a}\bar{b}}}$ and $\beta^{II}_{M^2_{a\bar{b}}}$ are more complicated, but the index structure fixes the possible terms that can appear. As before, the terms with different index structures must separately vanish.

Due to the symmetry imposed parameter relation $c_{ab} M^2_{\bar{a}\bar{b}} = 0$, we find that among the relations obtained from $\beta^{II}_{M^2_{\bar{a}\bar{b}}}$,

$$c_{ab} \big(\Lambda_{\bar{a}\bar{d}\bar{e}\bar{f}} \Lambda_{cdef} M^2_{\bar{c}\bar{b}} + \Lambda_{\bar{b}\bar{d}\bar{e}\bar{f}} \Lambda_{cdef} M^2_{\bar{c}\bar{a}} \big) \big|_{\rm sym} = 0 \,,$$

$$c_{ab} M^2_{cd} \Lambda_{\bar{a}\bar{c}\bar{e}\bar{f}} \Lambda_{ef\bar{d}\,\bar{b}} \big|_{\rm sym} = 0 \,.$$

CP symmetry yields $\Lambda_{\bar{a}\bar{b}\bar{c}\bar{d}} = \Lambda_{abcd}$ and $M^2_{\bar{a}\bar{b}} = M^2_{ab}$. The two displayed equations above hold for any choice of Λ_{abcd} and $\Lambda_{ab\bar{c}\bar{d}}$, subject to the parameter relations of the complexified theory:

$$\Lambda_{1111} = \Lambda_{2222} \,, \qquad \Lambda_{1112} = -\Lambda_{1222} \,,$$

$$\Lambda_{11\bar{1}\bar{1}} = \Lambda_{22\bar{2}\bar{2}} \,, \qquad \Lambda_{11\bar{1}\bar{2}} = -\Lambda^*_{12\bar{2}\bar{2}} \,, \qquad \Lambda_{11\bar{2}\bar{2}} = \Lambda^*_{11\bar{2}\bar{2}} \,,$$

$$c_{ab}\big(\Lambda_{\bar{a}\bar{d}\bar{e}\bar{f}}\Lambda_{cdef}M^2_{\bar{c}\bar{b}}+\Lambda_{\bar{b}\bar{d}\bar{e}\bar{f}}\Lambda_{cdef}M^2_{\bar{c}\bar{a}}\big)\Big|_{\rm sym}=0\,,$$
$$c_{ab}M^2_{cd}\Lambda_{\bar{a}\bar{c}\bar{e}\bar{f}}\Lambda_{ef\bar{d}\,\bar{b}}\Big|_{\rm sym}=0\,.$$

Since $\Lambda_{\bar{a}\bar{b}\bar{c}\bar{d}}$ and $\Lambda_{ab\bar{c}\bar{d}}$ are independent tensors (with compatible relations imposed by the symmetries), the above equations must hold if we numerically set $\Lambda_{ab\bar{c}\bar{d}}=\Lambda_{\bar{a}\bar{b}\bar{c}\bar{d}}$. With this choice, the equations above are algebraically equivalent to the corresponding equations of the original real scalar field theory!

Moreover, the same argument extends to arbitrary order in perturbation theory.

The origin of RG-stable parameter relations

Schematically, $\beta_{c_{ab}M^2_{\bar{a}\bar{b}}} = c_{ab}\sum_k f_k(M^2,\Lambda)_{\bar{a}\bar{b}}$.

Each term in the sum contains one factor of M^2 and n factors of Λ at order n, where M^2 can have index structure cd, $\bar{c}\bar{d}$, or $c\bar{d}$, and Λ can have index structure $cdef$, $cde\bar{f}$, $cd\bar{e}\bar{f}$, $c\bar{d}\bar{e}\bar{f}$, or $\bar{c}\bar{d}\bar{e}\bar{f}$. The indices must combine (including Kronecker deltas) such that the index structure of the f_k is $\bar{a}\bar{b}$. Then,

$$\beta_{c_{ab}M^2_{\bar{a}\bar{b}}}\Big|_{\rm sym} = 0 \qquad \Longrightarrow \qquad c_{ab}f_k(M^2,\Lambda)_{\bar{a}\bar{b}}\Big|_{\rm sym} = 0\,,$$

for each k separately.

There will always be at least one value of k where $f_k(M^2,\Lambda)$ depends on tensors with an even number of unbarred and barred indices, respectively. Since $\Lambda_{\bar{a}\bar{b}\bar{c}\bar{d}}$ and $\Lambda_{ab\bar{c}\bar{d}}$ are independent, we can numerically set $\Lambda_{ab\bar{c}\bar{d}} = \Lambda_{\bar{a}\bar{b}\bar{c}\bar{d}}$. Thus, for some value of k,

$$c_{ab} f_k(M^2,\Lambda)_{\bar{a}\bar{b}}\Big|_{\rm sym} = 0\,,$$

where the distinction between unbarred and barred indices can be neglected.[12] This equation will be algebraically equivalent to the corresponding equation of the original real scalar field theory.

> That is, the RG-stability of the parameter relations of the original real scalar field theory is inherited from the symmetry of the corresponding complexified theory.

[12] Having imposed CP symmetry, all scalar potential parameters are real.

The toy model with Yukawa couplings

Consider the coupling of the complex scalar field Φ of the toy model with a two-component fermion field ψ,

$$\mathscr{L} = \partial_\mu\Phi\partial^\mu\Phi^* - m_2^2\Phi^2 - m_2^{2*}\Phi^{*2} - \lambda_1(\Phi^*\Phi)^2 - \lambda_2\Phi^4 - \lambda_2^*\Phi^{*4}$$
$$-y\,\Phi\psi\psi - y^*\Phi^*\psi^\dagger\psi^\dagger\,,$$

after imposing the extended GOOFy symmetries $\Phi \to \Phi$, $\Phi^* \to -\Phi^*$, $\psi\psi \to \psi\psi$, $\psi^\dagger\psi^\dagger \to -\psi^\dagger\psi^\dagger$, which forbids the terms $m_1^2\Phi^*\Phi + \big(\lambda_3\Phi^2 + \lambda_3^*\Phi^{*2}\big)\Phi^*\Phi$ (i.e., $m_1^2 = \lambda_3 = 0$). Including the Yukawa terms in the RGEs of m_1^2 and λ_3, we again see that the relations $m_1^2 = \lambda_3 = 0$ are RG-stable.

With $\Phi = (\varphi_1 + i\varphi_2)/\sqrt{2}$, the GOOFy symmetry transformations of the scalars are $\varphi_1 \to i\varphi_2$ and $\varphi_2 \to -i\varphi_1$ as before.

Complexification promotes $\varphi_{1,2}$ to the complex fields $\Phi_{1,2}$ and doubles the number of two-component fermion fields. Imposing the conventional symmetries:[13]

$$\Phi_1 \to i\Phi_2\,, \quad \Phi_2 \to -i\Phi_1\,, \quad \psi_1\psi_1 \to i\psi_2\psi_2\,, \quad \psi_2\psi_2 \to -i\psi_1\psi_1\,.$$

yields the Yukawa Lagrangian of the complexified model:

$$-\mathscr{L}_{\rm Y} = y_1(\Phi_1\psi_1\psi_1 - \Phi_2\psi_2\psi_2) + y_2(\Phi_1\psi_2\psi_2 + \Phi_2\psi_1\psi_1) + \text{h.c.}$$

Conjecture: The RG-stability of the parameter relations of the original theory is still inherited from the symmetry of the corresponding complexified theory.

[13]The transformations of the individual fermions are given by $\psi_1 \to e^{i\pi/4}\psi_2$ and $\psi_2 \to e^{-i\psi/4}\psi_1$.

Future directions

1. Verify that RG-stability of the parameter relations of the toy model with Yukawa couplings is guaranteed by the conventional symmetries of the corresponding complexified model.

2. Construct new examples of RG-stable parameter relations in the absence of a symmetry starting from a complexified model.

3. Apply the results of this talk to the parameter relations of the 2HDM imposed by the GOOFy symmetries of Ferreira et al.

4. Are there more sophisticated ideas that can provide a deeper understanding of the phenomena explored in this work? (See Trautner's talk at this meeting.)

Final (wild) speculations

1. Spontaneous symmetry breaking via a GOOFy symmetry?

In the 2HDM with the GOOFy parameter relation $m_{22}^2 = -m_{11}^2$,

$$\mathcal{V} = m_{11}^2 \Phi_1^\dagger \Phi_1 + m_{22}^2 \Phi_2^\dagger \Phi_2 - [m_{12}^2 \Phi_1^\dagger \Phi_2 + \text{h.c.}] + \mathcal{V}_4\,,$$

the scalar potential exhibits spontaneous symmetry breaking independently of the numerical values of the parameters (assumed to be nonzero).

2. A natural hierarchy via a (slightly broken) GOOFy symmetry?[14]

Recall the toy model of one complex scalar field:

$$\mathcal{V} = m_1^2 \Phi^* \Phi + m_2^2 \Phi^2 + m_2^{2*} \Phi^{*2} + \mathcal{V}_4\,.$$

The GOOFy symmetry imposes $m_1^2 = 0$ (with no condition on m_2^2). Perhaps a slight breaking of this symmetry could yield $0 < m_1^2 \ll |m_2^2|$ naturally?

[14] See, e.g., T. de Boer, F. Goertz, and A. Incrocci, arXiv:2507.22111; A. Trautner, arXiv:2508.02646.

Backup slides

Symmetries of the 2HDM scalar potential

Consider the bosonic sector of the 2HDM:

$$\mathscr{L} = \mathscr{L}_{\rm KE} - \mathcal{V}(\Phi_1, \Phi_2)\,,$$

where $\mathscr{L}_{\rm KE} \equiv (D^\mu \Phi_a)^\dagger D_\mu \Phi_a$ (summed implicitly over $a = 1, 2$) is written in terms of the SU(2)×U(1)$_Y$ covariant derivative D_μ, and the scalar potential is given by:

$$\begin{aligned}
\mathcal{V} = {} & m_{11}^2 \Phi_1^\dagger \Phi_1 + m_{22}^2 \Phi_2^\dagger \Phi_2 - [m_{12}^2 \Phi_1^\dagger \Phi_2 + {\rm h.c.}] \\
& + \tfrac{1}{2}\lambda_1 (\Phi_1^\dagger \Phi_1)^2 + \tfrac{1}{2}\lambda_2 (\Phi_2^\dagger \Phi_2)^2 + \lambda_3 (\Phi_1^\dagger \Phi_1)(\Phi_2^\dagger \Phi_2) + \lambda_4 (\Phi_1^\dagger \Phi_2)(\Phi_2^\dagger \Phi_1) \\
& + \left\{ \tfrac{1}{2}\lambda_5 (\Phi_1^\dagger \Phi_2)^2 + \big[\lambda_6 (\Phi_1^\dagger \Phi_1) + \lambda_7 (\Phi_2^\dagger \Phi_2)\big] \Phi_1^\dagger \Phi_2 + {\rm h.c.} \right\} ,
\end{aligned}$$

where m_{11}^2, m_{22}^2, and $\lambda_1, \cdots, \lambda_4$ are real and m_{12}^2, λ_5, λ_6 and λ_7 are potentially complex parameters.

Parameter relations of the scalar potential can be the result of a symmetry that preserves the form of $\mathscr{L}_{\rm KE}$. Two classes of symmetries are possible:

(1) Higgs flavor (HF) symmetries: $\Phi_a \to S_{ab}\Phi_b$.

(2) generalized CP symmetries (GCP): $\Phi_a \to X_{ab}\Phi_b^*$.

All possible inequivalent symmetries of the 2HDM scalar potential have been classified and the corresponding parameter relations elucidated.[15] We do not distinguish between different symmetries that yield the same parameter relations.

[15] I.P. Ivanov, Phys. Lett. B **632** (2006) 360, hep-ph/0507132; Phys. Rev. D **75** (2007) 035001, hep-ph/0609018; P.M. Ferreira, H.E. Haber, and J.P. Silva, Phys. Rev. D **79** (2009) 116004, arXiv:0902.1537 [hep-ph]; P.M. Ferreira et al., Int. J. Mod. Phys. A **26** (2011) 769, arXiv:1010.0935 [hep-ph].

(1) HF symmetries [subgroups of U(2)]

$$\mathbb{Z}_2 : \Phi_1 \to \Phi_1\,, \quad \Phi_2 \to -\Phi_2\,,$$

$$\mathrm{U}(1) : \Phi_1 \to \Phi_1\,, \quad \Phi_2 \to e^{i\theta}\Phi_2\,, \quad 0 < \theta < 2\pi\,,$$

$$\mathrm{U}(2)/\mathrm{U}(1)_Y : \Phi_a \to S_{ab}\Phi_b\,, \quad \text{with } S \in \mathrm{U}(2)/\mathrm{U}(1)_Y\,.$$

(2) GCP symmetries

$$\mathrm{GCP1}: \;\; \Phi_1 \to \Phi_1^*\,, \quad \Phi_2 \to \Phi_2^*\,,$$

$$\mathrm{GCP2}: \;\; \Phi_1 \to \Phi_2^*\,, \quad \Phi_2 \to -\Phi_1^*\,,$$

$$\mathrm{GCP3}: \begin{cases} \Phi_1 \to \Phi_1^* \cos\theta + \Phi_2^* \sin\theta \\ \Phi_2 \to \Phi_2^* \cos\theta - \Phi_1^* \sin\theta \end{cases}, \quad 0 < \theta < \tfrac{1}{2}\pi\,.$$

In the case of GCP3, any choice of $0 < \theta < \frac{1}{2}\pi$ imposes the same conditions on the scalar potential parameters.

If we now impose the symmetries listed above in the scalar field basis $\{\Phi_1, \Phi_2\}$, we obtain the parameter relations listed below.

symmetry	m_{22}^2	m_{12}^2	λ_2	λ_4	$\mathrm{Re}\,\lambda_5$	$\mathrm{Im}\,\lambda_5$	λ_6	λ_7
$\mathbb{Z}_2$		0					0	0
U(1)		0			0	0	0	0
U(2)/U(1)$_Y$	m_{11}^2	0	λ_1	$\lambda_1 - \lambda_3$	0	0	0	0
GCP1		real				0	real	real
GCP2	m_{11}^2	0	λ_1					$-\lambda_6$
GCP3	m_{11}^2	0	λ_1		$\lambda_1 - \lambda_3 - \lambda_4$	0	0	0

Empty entries above correspond to a lack of constraints on the corresponding parameters.

What about other possible symmetries? For example,

$$\Pi_2 : \Phi_1 \longleftrightarrow \Phi_2\,,$$

$$\Pi_2' : \Phi_1 \to \Phi_2\,, \quad \Phi_2 \to -\Phi_1\,,$$

$$\mathrm{GCP1}' : \Phi_1 \to \Phi_2^*\,, \quad \Phi_2 \to \Phi_1^*\,,$$

symmetry	m_{22}^2	m_{12}^2	λ_2	$\mathrm{Re}\,\lambda_5$	$\mathrm{Im}\,\lambda_5$	λ_6	λ_7
Π_2	m_{11}^2	real	λ_1		0		λ_6^*
$\mathbb{Z}_2 \otimes \Pi_2$	m_{11}^2	0	λ_1		0	0	0
U(1)$\otimes\Pi_2$	m_{11}^2	0	λ_1	0	0	0	0
Π_2'	m_{11}^2	pure imaginary	λ_1		0		$-\lambda_6^*$
U(1)$'$	m_{11}^2	pure imaginary	λ_1	$\lambda_1 - \lambda_3 - \lambda_4$	0	0	0
U(1)$''$	m_{11}^2	real	λ_1	$\lambda_3 + \lambda_4 - \lambda_1$	0	0	0
GCP1$'$	m_{11}^2		λ_1				λ_6
GCP3$'$	m_{11}^2	0	λ_1	$\lambda_3 + \lambda_4 - \lambda_1$	0	0	0

Taken from H.E. Haber and J.P. Silva, Phys. Rev. D **103** (2021) 115012, arXiv:2102.07136 [hep-ph].

By a change of the scalar field basis, $\Phi_a \to U_{ab}\Phi_b$,[16] each of the symmetries above is equivalent to one of the six symmetries of the previous table (with its corresponding parameter relations).

For example, GCP1$'$ is equivalent to GCP1 in another basis even though GCP1$'$ (unlike GCP1) does not enforce reality conditions on the potentially complex parameters m_{12}^2, λ_5, λ_6, and λ_7.

[16]Here, $U \in \mathrm{U}(2)$ is the most general transformation that preserves the gauge-kinetic energy terms.

One-loop RGEs of the 2HDM

Note the following one-loop beta functions of the 2HDM:[17]

$$16\pi^2\beta_{m_{11}^2} = 3\lambda_1 m_{11}^2 + (2\lambda_3+\lambda_4)\, m_{22}^2 - 3\big(\lambda_6^* m_{12}^2 + \lambda_6 m_{12}^{2\,*}\big)\,,$$

$$16\pi^2\beta_{m_{22}^2} = (2\lambda_3+\lambda_4)\, m_{11}^2 + 3\lambda_2 m_{22}^2 - 3\big(\lambda_7^* m_{12}^2 + \lambda_7 m_{12}^{2\,*}\big)\,,$$

$$16\pi^2\beta_{m_{12}^2} = -3\left(\lambda_6 m_{11}^2 + \lambda_7 m_{22}^2\right) + (\lambda_3+2\lambda_4)\, m_{12}^2 + 3\lambda_5 m_{12}^{2\,*}\,,$$

$$16\pi^2\beta_{\lambda_1} = 6\lambda_1^2 + 2\lambda_3^2 + 2\lambda_3\lambda_4 + \lambda_4^2 + |\lambda_5|^2 + 12\,|\lambda_6|^2\,,$$

$$16\pi^2\beta_{\lambda_2} = 6\lambda_2^2 + 2\lambda_3^2 + 2\lambda_3\lambda_4 + \lambda_4^2 + |\lambda_5|^2 + 12\,|\lambda_7|^2\,,$$

$$16\pi^2\beta_{\lambda_5} = (\lambda_1+\lambda_2+4\lambda_3+6\lambda_4)\,\lambda_5 + 5\left(\lambda_6^2+\lambda_7^2\right) + 2\lambda_6\lambda_7\,,$$

$$16\pi^2\beta_{\lambda_6} = (6\lambda_1+3\lambda_3+4\lambda_4)\,\lambda_6 + (3\lambda_3+2\lambda_4)\,\lambda_7 + 5\lambda_5\lambda_6^* + \lambda_5\lambda_7^*\,,$$

$$16\pi^2\beta_{\lambda_7} = (6\lambda_2+3\lambda_3+4\lambda_4)\,\lambda_7 + (3\lambda_3+2\lambda_4)\,\lambda_6 + 5\lambda_5\lambda_7^* + \lambda_5\lambda_6^*\,.$$

[17] Here, the m_{ij}^2 and the λ_i are squared-mass and self-couplings parameters of the 2HDM scalar potential. The contributions of the gauge and Yukawa couplings have been neglected.

Some technical details

The formulae for the one-loop and two-loop beta functions, $\beta \equiv \beta^I + \beta^{II}$, of a real scalar field theory are:

$$\beta^I_{m^2_{ij}} = m^2_{mn}\lambda_{ijmn}\,,$$

$$\beta^I_{\lambda_{ijk\ell}} = \frac{1}{8}\sum_{\rm perm}\lambda_{ijmn}\lambda_{mnk\ell} = \lambda_{ijmn}\lambda_{mnk\ell} + \lambda_{ikmn}\lambda_{mnj\ell} + \lambda_{i\ell mn}\lambda_{mnjk}\,,$$

with an implicit sum over the repeated indices, where $\sum_{\rm perm}$ denotes a sum over the permutations of the uncontracted indices, i, j, k, and ℓ, and

$$\beta^{II}_{m^2_{ij}} = \frac{1}{12}\big(\lambda_{ik\ell m}\lambda_{nk\ell m}m^2_{nj} + \lambda_{jk\ell m}\lambda_{nk\ell m}m^2_{ni}\big) - 2m^2_{k\ell}\lambda_{ikmn}\lambda_{j\ell mn}\,,$$

$$\beta^{II}_{\lambda_{ijk\ell}} = \frac{1}{72}\sum_{\rm perm}\lambda_{inpq}\lambda_{mnpq}\lambda_{mjk\ell} - \frac{1}{4}\sum_{\rm perm}\lambda_{ijmn}\lambda_{kmpq}\lambda_{\ell npq}\,.$$

The β^{II} above each consist of the sum of two linearly independent combinations of tensor quantities. Each individual combination separately vanishes when the parameter relations (indicated by "sym") are applied.

Future directions (in more detail)

1. A recipe to create real scalar field theories with RG-stable parameter relations in absence of a symmetry:

- Start with a theory of n complex scalars Φ_a with RG-stable parameter relations due to some Higgs-Flavor (HF) symmetry [which is a subgroup of U(n), the symmetry group of $\mathscr{L}_{\rm KE}$].

- Impose CP to ensure the reality of all scalar potential parameters.

- Retain only the terms of the scalar potential that are holomorphic in the complex fields, i.e., of the form $M^2_{\bar{a}\bar{b}}\Phi_a\Phi_b + \Lambda_{\bar{a}\bar{b}\bar{c}\bar{d}}\Phi_a\Phi_b\Phi_c\Phi_d$.

- Construct the corresponding theory of n real scalars with the following recipe.
  - Replace the Φ_a with n real scalar fields φ_a.
  - Replace $\mathscr{L}_{\rm KE}$ with a canonically normalized kinetic energy term for the real scalar theory [with symmetry group O(n)].

- If the HF symmetry of the complex scalar field theory cannot be embedded in O(n), then this HF symmetry will not survive as a conventional symmetry of the real scalar field theory.

However, the symmetry-imposed parameter relations satisfied by $M_{\bar{a}\bar{b}}$ and $\Lambda_{\bar{a}\bar{b}\bar{c}\bar{d}}$ are now parameter relations of the same form in the resulting real scalar field theory. These parameter relations are RG-stable due to the symmetries of the complexified theory.

2. Applying the results of this talk to the GOOFy symmetries of the 2HDM.

- Starting from the 2HDM Lagrangian written in terms of complex doublets, perform the realification procedure to rewrite the theory in terms of eight real scalar fields.

$$\Phi_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} \varphi_1 + i\varphi_2 \\ \varphi_3 + i\varphi_4 \end{pmatrix}, \qquad \Phi_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} \varphi_5 + i\varphi_6 \\ \varphi_7 + i\varphi_8 \end{pmatrix},$$

- The GOOFy symmetries take the form

$$\varphi_1 \to i\varphi_6\,, \quad \varphi_2 \to i\varphi_5\,, \quad \varphi_3 \to i\varphi_8\,, \quad \varphi_4 \to i\varphi_7\,,$$

$$\varphi_5 \to -i\varphi_2\,, \quad \varphi_6 \to -i\varphi_1\,, \quad \varphi_7 \to -i\varphi_4\,, \quad \varphi_8 \to -i\varphi_3\,.$$

- Complexify the theory by promoting the φ_i to eight complex scalar fields Φ_a.

- Impose CP symmetry to ensure that all scalar potential parameters are real.

- Verify that the RG-stability of the 2HDM with $m_{11}^2 = -m_{22}^2$; $\lambda_1 = \lambda_2$; $\lambda_6 = -\lambda_7$ can be explained by the symmetries of the complexified theory. This step will require an extension of the techniques of this work to the gauge and Yukawa sectors.

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