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LIE SYMMETRY ANALYSIS OF THE FIELD EQUATIONS IN MULTI-HIGGS MODELS

Based on work in preparation

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Point symmetries of systems of PDEs

- System of m order n PDEs:

$$\Delta^i(x, y, y^{(1)}, \dots, y^{(n)}) = 0, \quad i \in \{1, \dots, m\} \tag{1}$$

where $y^{(j)}$ is the vector all j'th order derivatives.

- Point symmetry: Smooth transformation $S : \mathbb{R}^{d+q} \to \mathbb{R}^{d+q}$

$$S[(x, y)] = (\hat{x}, \hat{y}) \tag{2}$$

which maps solutions of $\Delta = 0$ to new solutions, i.e. preserve the structure of Δ:

$$\Delta = 0 \Rightarrow \hat{\Delta} \equiv \Delta(\hat{x}, \hat{y}, \hat{y}^{(1)}, \dots, \hat{y}^{(n)}) = 0 \tag{3}$$

with

$$\hat{y}_{,\mu_1 \dots \mu_k} \equiv S\left(\frac{d^k y^i}{dx^{\mu_1} \dots dx^{\mu_k}}\right) = \frac{d^k \hat{y}^i}{d\hat{x}^{\mu_1} \dots d\hat{x}^{\mu_k}}. \tag{4}$$

Field equations

- Want to let Δ be the field eqs. of a theory with Lagrangian $\mathcal{L}$ and action $\mathcal{S}$

$$\mathcal{S} = \int \mathcal{L} dx^0 \cdots dx^{d-1}. \tag{5}$$

- Consider variations of the fields y, demand stationary values of $\mathcal{S} \Rightarrow$ Euler-Lagrange equations i.e. field equations

$$E_i(\mathcal{L}) = 0, \quad E_i = \frac{\partial}{\partial y^i} - d_\mu \frac{\partial}{\partial y^i_{,\mu}} + \ldots, \qquad i \in \{1, \ldots, q\}, \tag{6}$$

with $d_\mu \equiv d/dx^\mu$, the total derivative.

3 kinds of symmetries of the field equations

- Strict variational symmetries: First order variation

$$\delta(\mathcal{L} dx^0 \cdots dx^{d-1}) = 0 \tag{i}$$

- Divergence symmetries:

$$\delta(\mathcal{L} dx^0 \cdots dx^{d-1}) = d_\mu \beta^\mu \tag{ii}$$

- (i) and (ii) are variational symmetries, i.e. symmetries of the action $\mathcal{S}$.
- (iii): Non-variational symmetries, only symmetries of $E(\mathcal{L}) = 0$
- $\Rightarrow$ 3 Lie symmetry algebras

$$\mathfrak{g}_{\text{svar}} \subseteq \mathfrak{g}_{\text{var}} \subseteq \mathfrak{g}_{\text{EL}} \tag{7}$$

Finding the Lie point symmetries

- Lie point symmetry: A family point symmetries $\exp(tX)$ smoothly connected to identity (i.e. not discrete)
- Infinitesimal generator of a point transformation is a vector field

$$X = \xi^{\mu}(x,y)\frac{\partial}{\partial x^{\mu}} + \eta^{i}(x,y)\frac{\partial}{\partial y^{i}}, \tag{8}$$

- From now on: $\xi = 0$ (not considering spacetime trafos). Want to determine $\eta^{i}(x,y)$ that generates symmetries.
- Have to consider the prolongation of X to derivatives:

$$\mathrm{pr}\, X = X + (d_{\mu}\eta^{i})\frac{\partial}{\partial y^{i}_{,\mu}} + \sum_{\mu\leq\nu}(d_{\mu}d_{\nu}\eta^{i})\frac{\partial}{\partial y^{i}_{,\mu\nu}} + \dots \tag{9}$$

- Hence, if $X(\phi_1) = \phi_2$ then $\mathrm{pr}\, X(\partial_{\mu}\phi_1) = \partial_{\mu}\phi_2$. Moreover,

$$\mathrm{pr}\, X(\mathcal{L}) = \delta\mathcal{L}. \tag{10}$$

The determining equations

- Lie: (holds for general Δ): $\mathfrak{g}$ is a symmetry algebra of $E(\mathcal{L}) = 0$ if and only if

$$(\mathrm{pr}\, X(E_i(\mathcal{L})))|_{E(\mathcal{L})=0} = 0 \quad \forall i \in \{1, \ldots, q\}, \tag{11}$$

  for all infinitesimal generators $X \in \mathfrak{g}$.
- (11) is called the *determining equations* $\mathcal{D}_j = 0$ of $\mathfrak{g}$.
- $\mathcal{D}$ large, over-determined system of linear PDEs in the coefficients $\xi^\mu(x, y)$ and $\eta^i(x, y)$
- We use the `Mathematica` package `SYM` to find $\mathcal{D}$.[1]

[1] Dimas, S. and Tsoubelis, D. SYM: A new symmetry-finding package for Mathematica

Two-Higgs-doublet model (2HDM)

- Test our understanding and implementation of Lie's method by applying to a well-understood example: 2HDM:

$$\mathcal{L} = -\frac{1}{4} W^a_{\mu\nu} W^{a\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} + \sum_{i=1}^{2} (D_\mu \Phi_i)^\dagger (D^\mu \Phi_i) - V(\Phi_1, \Phi_2), \tag{12}$$

with Higgs doublets

$$\Phi_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_1 + i\phi_2 \\ \phi_3 + i\phi_4 \end{pmatrix}, \Phi_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_5 + i\phi_6 \\ \phi_7 + i\phi_8 \end{pmatrix} \tag{13}$$

and most general potential

$$\begin{aligned} V(\Phi_1, \Phi_2) = {} & m_{11}^2 \, \Phi_1^\dagger \Phi_1 + m_{22}^2 \, \Phi_2^\dagger \Phi_2 - \left[m_{12}^2 \, \Phi_1^\dagger \Phi_2 + \text{h.c.} \right] \\ & + \frac{\lambda_1}{2} (\Phi_1^\dagger \Phi_1)^2 + \frac{\lambda_2}{2} (\Phi_2^\dagger \Phi_2)^2 + \lambda_3 (\Phi_1^\dagger \Phi_1)(\Phi_2^\dagger \Phi_2) + \lambda_4 (\Phi_1^\dagger \Phi_2)(\Phi_2^\dagger \Phi_1) \\ & + \left[\frac{\lambda_5}{2} (\Phi_1^\dagger \Phi_2)^2 + \lambda_6 (\Phi_1^\dagger \Phi_1)(\Phi_1^\dagger \Phi_2) + \lambda_7 (\Phi_2^\dagger \Phi_2)(\Phi_1^\dagger \Phi_2) + \text{h.c.} \right]. \end{aligned} \tag{14}$$

- 2HDM: 24 Euler-Lagrange equations (16 gauge fields + 8 real, scalar fields)

$$E_i(\mathcal{L}) = 0, \quad 1 \le i \le 24 \quad \overset{\text{SYM}}{\Rightarrow} \quad \mathcal{D}_j = 0 \tag{15}$$

- We will only consider (purely) scalar symmetries, and set:

$$\begin{aligned} \xi^{\mu} &= 0, \quad \text{for all} \quad 0 \le \mu \le 3, \\ \eta^{i} &= 0, \quad \text{for all} \quad 9 \le i \le 24. \end{aligned} \tag{16}$$

- $\Rightarrow$ Simplest determining equations:

$$\partial_{\phi_j} \partial_{\phi_k} \eta^{i} = 0, \quad \text{for all} \quad 1 \le i, j, k \le 8, \tag{17}$$

- $\Rightarrow$ only affine scalar symmetries are possible:

$$\eta^{i} = a_i + b_{ij} \phi_j, \quad 1 \le i, j \le 8, \tag{18}$$

and we substitute (18) into the determining eqs.+demand all coefficients of all distinct monomials in the fields are zero $\Rightarrow \mathcal{D}'_k = 0$.

Solutions of the determining equations

- By considering different parameter cases (like $m_{11}^2 \neq m_{22}^2$) and reducing the potential (like $m_{12}^2 = \Im(\lambda_5) = 0$) we can solve for the a_i's (= 0) and the b_{ij}'s.
- The different solutions correspond to symmetry algebras and generators

$$\mathfrak{su}(2)_{\mathrm{HF}} \oplus \mathfrak{u}(1)_Y = \mathrm{span}(H_1, H_2, X_{\mathrm{PQ}}, X_Y), \tag{19}$$

$$\text{Equivalent} \begin{cases} \mathfrak{u}(1)_1 \oplus \mathfrak{u}(1)_Y = \mathrm{span}(H_1, X_Y), & (20) \\ \mathfrak{u}(1)_2 \oplus \mathfrak{u}(1)_Y = \mathrm{span}(H_2, X_Y), & (21) \\ \mathfrak{u}(1)_3 \oplus \mathfrak{u}(1)_Y = \mathrm{span}(X_{\mathrm{PQ}}, X_Y), & (22) \end{cases}$$

$$\mathfrak{u}(1)_Y = \mathrm{span}(X_Y). \tag{23}$$

for generators

$$H_1 = \phi_6\partial_{\phi_1} - \phi_5\partial_{\phi_2} + \phi_8\partial_{\phi_3} - \phi_7\partial_{\phi_4} + \phi_2\partial_{\phi_5} - \phi_1\partial_{\phi_6} + \phi_4\partial_{\phi_7} - \phi_3\partial_{\phi_8}, \tag{24}$$

$$H_2 = \phi_5\partial_{\phi_1} + \phi_6\partial_{\phi_2} + \phi_7\partial_{\phi_3} + \phi_8\partial_{\phi_4} - \phi_1\partial_{\phi_5} - \phi_2\partial_{\phi_6} - \phi_3\partial_{\phi_7} - \phi_4\partial_{\phi_8}, \tag{25}$$

$$X_{\mathrm{PQ}} = \phi_2\partial_{\phi_1} - \phi_1\partial_{\phi_2} + \phi_4\partial_{\phi_3} - \phi_3\partial_{\phi_4} - \phi_6\partial_{\phi_5} + \phi_5\partial_{\phi_6} - \phi_8\partial_{\phi_7} + \phi_7\partial_{\phi_8}, \tag{26}$$

$$X_Y = -\phi_2\partial_{\phi_1} + \phi_1\partial_{\phi_2} - \phi_4\partial_{\phi_3} + \phi_3\partial_{\phi_4} - \phi_6\partial_{\phi_5} + \phi_5\partial_{\phi_6} - \phi_8\partial_{\phi_7} + \phi_7\partial_{\phi_8} \tag{27}$$

The possible, inequivalent symmetry algebras

- I.e. only three inequivalent symmetry algebras, $\mathfrak{su}(2)_{\mathrm{HF}} \oplus \mathfrak{u}(1)_Y$, $\mathfrak{u}(1) \oplus \mathfrak{u}(1)_Y$ and $\mathfrak{u}(1)_Y$, consistent with established theory.
- Moreover, we show that under "mild" conditions (satisfied by the 2HDM), for kinetic terms T and a symmetry X we generally have

$$\mathrm{pr}\, X(T) = 0 \Rightarrow X \text{ is strictly variational} \tag{28}$$

- $\Rightarrow$ all symmetries in the 2HDM are strictly variational, no matter the different parameter conditions.
- $\Rightarrow$ no divergence symmetries or non-variational symmetries in the 2HDM.

SM+S

- The Lie symmetry method is "universal" and may be applied to any model, e.g. the SM augmented by a real, scalar gauge singlet s:

$$\mathcal{L}_{\text{SMS}} = -\frac{1}{4}W^a_{\mu\nu}W^{a\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} + (D_\mu\Phi)^\dagger(D^\mu\Phi) + \frac{1}{2}\partial_\mu s\partial^\mu s - V(\Phi, s), \quad (29)$$

$$V(\Phi, s) = -\mu^2\Phi^\dagger\Phi + \lambda(\Phi^\dagger\Phi)^2 + \alpha s - \mu_s s^2 + \kappa s^3 + \lambda_s s^4 + \kappa_{s\phi}\Phi^\dagger\Phi s + \lambda_{s\phi}\Phi^\dagger\Phi s^2. \quad (30)$$

- Again we only find affine, scalar symmetries ($\phi_5 \equiv s$)

$$\eta^i = a_i + b_{ij}\phi_j, \quad 1 \le i, j \le 5. \quad (31)$$

- No reparametrization freedom, but a small subtlety concerning the linear term αs which usually, but not always, may be eliminated through a shift $s \to s - \beta$.
- We find the 4 possible symmetries (only X_Y and $X_1 = \partial_s$ variational),

$$\mathfrak{a}(1) \oplus \mathfrak{u}(1)_Y,\ \mathfrak{u}(1)_1 \oplus \mathfrak{u}(1)_Y,\ \mathfrak{u}(1)_2 \oplus \mathfrak{u}(1)_Y,\ \mathfrak{u}(1)_Y. \quad (32)$$

SM+2S

- Lagrangian of SM+KS may be written, with sums up to K:

$$\mathcal{L}_{\text{SMKS}} = -\frac{1}{4}W^a_{\mu\nu}W^{a\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} + (D_\mu\Phi)^\dagger(D^\mu\Phi) + \frac{1}{2}\partial_\mu s_i \partial^\mu s_i - V(\Phi, s), \quad (33)$$

$$\begin{aligned} V(\Phi, s) &= -\mu^2\Phi^\dagger\Phi + \lambda(\Phi^\dagger\Phi)^2 \\ &+ \alpha_i s_i - \sum_{i\leq j} m_{ij}s_is_j + \sum_{i\leq j\leq k}\kappa_{ijk}s_is_js_k + \sum_{i\leq j\leq k\leq l}\lambda_{ijkl}s_is_js_ks_l \\ &+ \kappa_i s_i\Phi^\dagger\Phi + \sum_{i\leq j}\lambda_{ij}s_is_j\Phi^\dagger\Phi. \end{aligned} \quad (34)$$

- $K = 2$: We only get affine, scalar symmetries ($\phi_5 \equiv s_1, \phi_6 \equiv s_2$)

$$\eta^i = a_i + b_{ij}\phi_j \quad \text{for} \quad 1 \leq i, j, k \leq 6. \quad (35)$$

- Have to keep the potential reduced, due to O(2) reparametrization freedom. Some subtleties concerning the linear terms $\alpha_i s_i$ which sometimes cannot be eliminated through a shift $s_j \to s_j - \beta_j$. (Work in progress.)

Conclusions

- Lie symmetry analysis is a "universal" method for finding all Lie symmetries (including scalar, gauge, spacetime and higher order symmetries).
- Missing discrete symmetries may be identified through the automorphism groups of the Lie symmetry algebras obtained through the method.
- May be applied to models with several parameters, several variables and reparametrization freedom like 2HDM and SM+2S.
- Detects all symmetries of the field equations of a model, including divergence and non-variational symmetries.
- Hence, there are no divergence or non-variational symmetries in the 2HDM.