Gravity and the Higgs Mass

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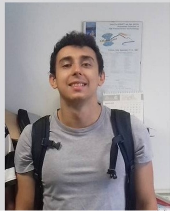

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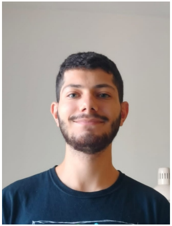

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C. Branchina, V. Branchina, F. Contino and A. Pernace, *Path integral measure and the cosmological constant*, Phys.Rev.D**111** (2025) no.10, 105018, arXiv:2412.10194.

C. Branchina, V. Branchina, F. Contino and A. Pernace, *Path integral measure and RG equations for gravity*, Phys.Rev.D**111** (2025) no.12, 125021, arXiv:2412.14108.

C. Branchina, V. Branchina, F. Contino, R. Gandolfo and A. Pernace, *On the RG flow of the Newton and cosmological constant*, arXiv:2505.07628.

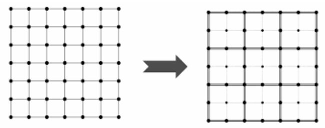
C. Branchina, V. Branchina, F. Contino, R. Gandolfo and A. Pernace, *Diffeomorphism invariance of the effective gravitational action*, Phys.Rev.D**112** (2025) no.4, 045002, arXiv:2506.05100.

C. Branchina, V. Branchina, F. Contino, R. Gandolfo and A. Pernace, *Gravity and the Higgs boson mass*, arXiv:2507.13832.

Renormalization

.. **Evergreen subject** .. source of **frustration** and **surprises** ..

Physical understanding: Wilson ,



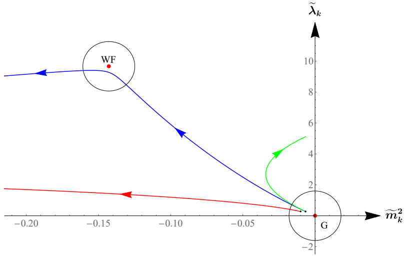
Theory at Λ: S_Λ $\rightarrow$ Theory at $\Lambda/2$: $S_{\Lambda/2}$ $\rightarrow$ ... $\rightarrow$ Γ

Progressive inclusion of fluctuations, physical running scale $\Lambda \rightarrow \Lambda/2 \rightarrow \Lambda/4 \rightarrow \Lambda/8 \rightarrow \ldots$

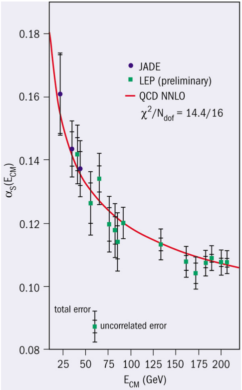
Piling up of fluctuations $\rightarrow$ Evolution of parameters

Renormalization : Universal language

$D = 3$ dimensions : Wilson-Fisher



$D = 4$ dimensions : AF



Renormalization: Gravity and CC problem

Quantum gravity: perturbatively non-renormalizable

- EFT valid up to the physical cutoff scale Λ ($\sim M_P$)
- Non-perturbatively renormalizable: UV fixed point

Contribution to vacuum energy **from quantum fluctuations** $\sim \Lambda^4$

CC problem: most severe naturalness problem

Several attempts towards its solution ... (here just a few examples ...)

Polyakov ... and later Jackiw ... Moscow zero ...

Coleman ... Wormholes

Taylor - Veneziano ... non-local terms : $V \log V$

... and many other attempts ...

Sometimes: SUSY invoked (SUGRA embedding)

Effective action $\Gamma_{\rm grav}$

Phys.Rev.D111 (2025) 10, 105018; arXiv:2412.10194

Phys.Rev.D111 (2025) 12, 125021; arXiv:2412.14108

Technical tools used in the analysis

- Gauge-invariant VDW one-loop effective action, $\Gamma^{1l}_{\rm grav} = S_{\rm grav} + \delta S^{1l}_{\rm grav}$
- Strategy put forward by Fradkin and Tseytlin / Taylor and Veneziano
- Background field method: $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$ ($\bar{g}_{\mu\nu}$ is the background)
- When $\bar{g}_{\mu\nu}$ has spherical symmetry, one-loop VDW effective action coincides with the standard one calculated with gauge-fixing term

$$S_{\rm gf} = \frac{1}{32\pi G\xi}\int d^4x\sqrt{\bar{g}}\left[\nabla_\mu\left(h^\mu_\nu - \frac{1}{2}\delta^\mu_\nu\, h^\sigma_\sigma\right)\right]$$

after taking the limit $\xi \to 0$ at the end of the calculation

Standard calculation: spherical background

Einstein-Hilbert Action : $S_{\rm grav} = \frac{1}{16\pi G} \int d^4x \sqrt{g}\, (-R + 2\Lambda)$

Take spherical background $\bar{g}_{\mu\nu} = g^{(a)}_{\mu\nu}$ $\left(\int d^4x \sqrt{g^{(a)}} = \frac{8\pi^2}{3} a^4 \;,\; R(g^{(a)}) = \frac{12}{a^2} \right)$

Einstein-Hilbert Action $S^{(a)}_{\rm grav} = \frac{\pi \Lambda_{\rm cc}}{3G} a^4 - \frac{2\pi}{G} a^2$ dS solution $a_{\rm dS} = \sqrt{\frac{3}{\Lambda_{\rm cc}}}$

One-loop VDW effective action = the standard one calculated with gauge-fixing term

$$S_{\rm gf} = \frac{1}{32\pi G \xi} \int d^4x \sqrt{\bar{g}} \left[\nabla_\mu \left(h^\mu_\nu - \frac{1}{2} \delta^\mu_\nu\, h^\sigma_\sigma \right) \right]$$

after taking the limit $\xi \to 0$ at the end of the calculation Fradkin,Tseytlin

Add to $S_{\rm grav} + S_{\rm gf}$ the corresponding ghost action (v_μ vector ghost fields)

$$S_{\rm ghost} = \frac{1}{32\pi G} \int d^4x \sqrt{g^{(a)}}\, g^{(a)\,\mu\nu}\, v^*_\mu \left(-\nabla_\rho \nabla^\rho - \frac{3}{a^2} \right) v_\nu$$

Standard calculation: spherical background

One-loop effective action $\Gamma^{1l}_{\rm grav} = S_{\rm grav} + \delta S^{1l}_{\rm grav}$

One-loop correction $\delta S^{1l}_{\rm grav}$ given by

$$e^{-\delta S^{1l}_{\rm grav}} = \lim_{\xi\to 0} \int [\mathcal{D}\mu]\, e^{-\delta S^{(2)}} \quad ; \quad \delta S^{(2)} \equiv S_2 + S_{\rm gf} + S_{\rm ghost}$$

S_2 quadratic term in the expansion of $S_{\rm grav}[\, g^{(a)}_{\mu\nu} + h_{\mu\nu}]$

$$S_2 \equiv \frac{1}{32\pi G} \int {\rm d}^4 x\, \sqrt{g^{(a)}} \left[\frac{1}{2} \widetilde{h}^{\mu\nu} \left(-\nabla_\rho \nabla^\rho - 2\Lambda_{\rm cc} + \frac{8}{a^2} \right) h_{\mu\nu} + \frac{h^2}{a^2} - \nabla^\rho \widetilde{h}_{\rho\mu} \nabla^\sigma \widetilde{h}^\mu_\sigma \right]$$

$h \equiv g^{(a)}_{\mu\nu} h^{\mu\nu}$, $\widetilde{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2} g^{(a)}_{\mu\nu} h$

indices raised with $g^{(a)\,\mu\nu}$; covariant derivatives in terms of $g^{(a)}_{\mu\nu}$

One-loop corrections to $\frac{\Lambda_{\rm cc}}{G}$ and $\frac{1}{G}$ given by coefficients of a^4 and a^2 in $\delta S^{1l}_{\rm grav}$

... Delicate point ... The Measure $[\mathcal{D}\mu]$

Path integral measure - Fradkin-Vilkovisky

$$\left[\mathcal{D}\mu\right] \equiv \prod_x \left[g^{(a)\,00}(x) \, \left(g^{(a)}(x)\right)^{-1} \left(\prod_{\alpha\leq\beta} \mathrm{d}h_{\alpha\beta}(x)\right) \left(\prod_\rho \mathrm{d}v^*_\rho(x)\right) \left(\prod_\sigma \mathrm{d}v_\sigma(x)\right)\right]$$

Liouville meas. in phase space $\Rightarrow$ integrate over conj. momenta $\Rightarrow$ $g^{(a)\,00}\left(g^{(a)}\right)^{-1}$

Recent discussion: g^{00} dangerous? Diffeomorphism invariance?

PRD 112 (2025) 045002, arXiv:2506.05100 / Bonanno,Falls,Ferrero, JHEP 05 (2025) 164

Now : $g^{(a)}_{\mu\nu} = a^2 \, \widetilde{g}_{\mu\nu}$ $\quad \widetilde{g}_{\mu\nu}$ metric of a sphere of unitary radius, $a = 1$

convenient (though not necessary) way to dig out the a-dependence from the background metric

Redefinition : $\widehat{h}_{\mu\nu} = (32\pi G)^{-1/2} \, a^{-1} h_{\mu\nu}$, $v_\mu \to (32\pi G)^{\frac{1}{2}} \, v_\mu$

convenient but not necessary (arXiv:2505.07628 / Held,Knorr,Pawlowski,Platania,Reichert, arXiv:2504.12006

$$\Longrightarrow \quad \left[\mathcal{D}\mu\right] \sim \prod_x \left[\left(\prod_{\alpha\leq\beta} \mathrm{d}\widehat{h}_{\alpha\beta}(x)\right)\left(\prod_\rho \mathrm{d}v^*_\rho(x)\right)\left(\prod_\sigma \mathrm{d}v_\sigma(x)\right)\right]$$

The radius “a” does not appear in the measure

$\delta S^{(2)} = S_2 + S_{\rm gf} + S_{\rm ghost}$ contains

only dimensionless operators

... All this boils down to ...

Dimensionless Laplace-Beltrami operator $-\widetilde{\Box}^{(s)} \equiv -a^2 \, \Box_a^{(s)}$

$-\Box_a^{(s)}$ Laplace-Beltrami for sphere of radius a; s spins: $s = 0, 1, 2$

Dimensionless eigenvalues $\lambda_n^{(s)}$ and corresponding degeneracies $D_n^{(s)}$

$$\lambda_n^{(s)} = n^2 + 3n - s \quad ; \quad D_n^{(s)} = \tfrac{2s+1}{3} \left(n + \tfrac{3}{2}\right)^3 - \tfrac{(2s+1)^3}{12} \left(n + \tfrac{3}{2}\right) \quad ; \quad n = s, s+1, \ldots$$

Expanding $\widehat{h}_{\mu\nu}$, v_ρ^* and v_σ for $\delta S^{1l}_{\rm grav}$

$$\delta S^{1l}_{\rm grav} = -\frac{1}{2} \log \frac{\det_1[-\widetilde{\Box}^{(1)} - 3] \det_2[-\widetilde{\Box}^{(0)} - 6]}{\det_0[-\widetilde{\Box}^{(2)} - 2a^2\Lambda + 8] \det_2[-\widetilde{\Box}^{(0)} - 2a^2\Lambda]} + \ldots$$

Laplacians of a **unit sphere** although we have a sphere of **radius** a

Now we have to calculate the determinants $\det[-\widetilde{\Box} + \alpha]$

Calculation of the "det" with two different strategies

Eigenvalues ; Proper-time

Calculation of "det's" - 1.Eigenvalues

$$\delta S^{1l}_{\text{grav}} = \frac{1}{2}\sum_{n=2}^{N-2}\left[D_n^{(2)}\log\left(\lambda_n^{(2)} - 2a^2\Lambda + 8\right) + D_n^{(0)}\log\left(\lambda_n^{(0)} - 2a^2\Lambda\right) - D_n^{(1)}\log\left(\lambda_n^{(1)} - 3\right) - D_n^{(0)}\log\left(\lambda_n^{(0)} - 6\right)\right] + \dots$$

N: numerical UV cutoff on the number of eigenvalues

De Sitter solution for the classical action $a_{\text{dS}} = \sqrt{\frac{3}{\Lambda_{\text{cc}}}}$

a_{dS} size of the universe $\implies$ $\Lambda \sim M_P = \frac{N}{a_{\text{dS}}} = N\sqrt{\frac{\Lambda_{\text{cc}}}{3}}$

Expanding for $N \gg 1$

$$\delta S^{1l}_{\text{grav}} = -\left(\Lambda_{\text{cc}}^2 \log N^2\right) a^4 + \Lambda_{\text{cc}}\left(-N^2 + 8\log N^2\right) a^2 + \frac{N^4}{24}\left(-1 + 2\log N^2\right) + \frac{N^2}{36}\left(203 - 75\log N^2\right) - \frac{779}{90}\log N^2$$

$$\delta S^{1l}_{\text{grav}} = -\left(\Lambda_{\text{cc}}^2 \log\frac{3\Lambda^2}{\Lambda_{\text{cc}}}\right) a^4 + \left(-3\Lambda^2 + 8\Lambda_{\text{cc}}\log\frac{3\Lambda^2}{\Lambda_{\text{cc}}}\right) a^2 + \dots$$

Calculation of the “det” - 2.Proper time

$(-\widetilde{\Box}^{(s)} - \alpha)$ dimensionless $\Longrightarrow$ $\det_i$ regularized with a dimensionless proper-time τ (lower cut: $N \gg 1$) with kernel kernel $\mathrm{K}_i^{(s)}(\tau)$

$$\det_i(-\Box^{(s)}_{a=1} - \alpha) = e^{-\int_{1/N^2}^{+\infty} \frac{d\tau}{\tau} \mathrm{K}_i^{(s)}(\tau)} \quad ; \quad \mathrm{K}_i^{(s)}(\tau) = \sum_{n=s+i}^{+\infty} D_n^{(s)}\, e^{-\tau\left(\lambda_n^{(s)} - \alpha\right)}$$

After integration over τ, sum over n with EML sum formula. Expanding for $N \gg 1$

$$\delta S^{1l}_{\mathrm{grav}} = -\left(\Lambda_{\mathrm{cc}}^2 \log N^2\right) a^4 + \Lambda_{\mathrm{cc}}\left(-N^2 + 8\log N^2\right) a^2 - \frac{N^4}{12} + \frac{17}{3}N^2 - \frac{1859}{90}\log N^2 + \dots$$

$$\text{With} \quad \Lambda \equiv \frac{N}{a_{\mathrm{dS}}} = \sqrt{\frac{\Lambda_{\mathrm{cc}}}{3}}\, N \quad \Longrightarrow$$

$$\delta S^{1l}_{\mathrm{grav}} = -\left(\Lambda_{\mathrm{cc}}^2 \log\frac{3\Lambda^2}{\Lambda_{\mathrm{cc}}}\right) a^4 + \left(-3\Lambda^2 + 8\Lambda_{\mathrm{cc}}\log\frac{3\Lambda^2}{\Lambda_{\mathrm{cc}}}\right) a^2 + \dots$$

The two methods give the same result

Coefficients of a^4 and a^2 : give one-loop corrections to $\frac{\Lambda_{cc}}{G}$ and $\frac{1}{G}$

$$\frac{\Lambda_{cc}^{1l}}{G^{1l}} = \frac{\Lambda_{cc}}{G}\left(1 - \frac{3G\Lambda_{cc}}{\pi}\log\frac{3\Lambda^2}{\Lambda_{cc}}\right) \quad ; \quad \frac{1}{G^{1l}} = \frac{1}{G}\left[1 + \frac{G}{2\pi}\left(3\Lambda^2 - 8\Lambda_{cc}\log\frac{3\Lambda^2}{\Lambda_{cc}}\right)\right]$$

Unexpected result: *only* logarithmic corrections to $\rho_{vac} = \frac{\Lambda_{cc}}{8\pi G}$

Taking for G the natural value $G \sim M_P^{-2}$ we see that quantum corrections do not spoil the naturalness of this relation

No naturalness problem with the renormalization of the Newton constant

$$G \sim G^{1l} \sim M_P^{-2}$$

Usual result: $\rho_{vac} \sim M_P^4 \implies$ bare value of $\rho_{vac} \sim M_P^4$ with coefficient to be enormously fine-tuned to cancel the one-loop M_P^4 correction

Our result: loop corrections $\rightarrow$ only mild (log) correction to ρ_{vac}

No Naturalness Problem in pure gravity

No need for Bare $\Lambda_{cc} \sim M_P^2$

We may naturally have $\Lambda_{cc} \ll M_P^2$

$$\Lambda_{cc}^{1l} \sim \Lambda_{cc}$$

... this result is unexpected ...

Usually we have power-like divergences in ρ_{vac}

Can we understand why we do not see power-like divergences?

Connect for a moment N and Λ through $\Lambda = \frac{N}{a}$ (rather than through $\Lambda = \frac{N}{a_{\rm dS}}$)

$$\delta S^{1l}_{\rm grav} = -\left(\Lambda^2_{\rm cc} \log N^2\right) a^4 + \Lambda_{\rm cc}\left(-N^2 + 8 \log N^2\right) a^2 - \frac{N^4}{12} + \frac{17}{3} N^2 - \frac{1859}{90} \log N^2 + \ldots$$

becomes

$$\delta S^{1l}_{\rm grav} = -\left[\frac{\Lambda^4}{12} + \Lambda_{\rm cc}\Lambda^2 + \Lambda^2_{\rm cc} \log\left(\Lambda^2 a^2\right)\right] a^4 + \left[\frac{17}{3}\Lambda^2 + 8\Lambda_{\rm cc} \log\left(\Lambda^2 a^2\right)\right] a^2 - \frac{1859}{90} \log\left(\Lambda^2 a^2\right)$$

spurious dependence on $g^{(a)}_{\mu\nu}$ in $\delta S^{1l}_{\rm grav}$

Conclusions - 1

In **pure gravity**

No quartic (Λ^4) and quadratic (Λ^2) divergences in vacuum energy when:

- identification of the physical cutoff properly done
- truly diffeomorphism invariant measure (FV) used

$\Longrightarrow$ No Naturalness Problem

(I stress that here we are considering only pure gravity)

Note - usual calculation (heat kernel expansion) in pure gravity:

quartic and quadratic divergences found

Let's move now to RG ...

Wilsonian Renormalization Group

UV action

$$S_{\rm grav}^{\rm UV}[\, g_{\mu\nu}] \equiv S_N[\, g_{\mu\nu}] = \frac{1}{16\pi G_N} \int \mathrm{d}^4 x \, \sqrt{g} \, (-R + 2\Lambda_N) \qquad (N \;\; \text{integer})$$

As before: physical cutoff $\Lambda_{\rm cut} \, (\sim M_P) = N/a_{\rm dS}$

Wilsonian action: $S_L[\, g_{\mu\nu}^{(a)}]$ $\qquad (L \text{ integer}, \; L < N \; ; \; \delta L \ll L)$

RG equation:

$$S_{L-\delta L}[\, g_{\mu\nu}^{(a)}] = S_L[\, g_{\mu\nu}^{(a)}] + \delta S_L \equiv S_L[\, g_{\mu\nu}^{(a)}] + \sum_{n=L-\delta L}^{L} f_L(n)$$

where

$$\delta S_L = -\frac{1}{2} \log \frac{\det_1[-\widetilde{\Box}^{(1)} - 3] \det_2[-\widetilde{\Box}^{(0)} - 6]}{\det_0[-\widetilde{\Box}^{(2)} - 2a^2\Lambda_L + 8] \det_2[-\widetilde{\Box}^{(0)} - 2a^2\Lambda_L]}$$

$$\Updownarrow$$

$$f_L = D_n^{(2)} \log \left(\lambda_n^{(2)} - 2a^2\Lambda_L + 8 \right) + D_n^{(0)} \log \left(\lambda_n^{(0)} - 2a^2\Lambda_L \right) - D_n^{(1)} \log \left(\lambda_n^{(1)} - 3 \right) - D_n^{(0)} \log \left(\lambda_n^{(0)} - 6 \right)$$

Computing the r.h.s. (**direct sum** or **proper time**), and expanding for $L \gg 1$:

$$L \frac{\partial}{\partial L} S_L = 2\Lambda_L^2 \, a^4 + 2\Lambda_L \left(L^2 - 8 \right) a^2 + \frac{L^4}{3} - \frac{34 L^2}{3} + \frac{1859}{45} + \mathcal{O}\left(\frac{1}{L^2} \right)$$

RG equations - physical running scale k

Physical running scale $$k \equiv \frac{L}{\bar{a}_L} = L\sqrt{\frac{\Lambda_L}{3}}$$ $\left(k_{\rm IR} \leq k \leq \Lambda_{\rm cut}\ ;\ k_{\rm IR} = (\tfrac{16}{3}\,\Lambda_4)^{1/2}\right)$

The RG equations become $(\Lambda_k \equiv \Lambda_L,\ G_k \equiv G_L)$

$$k\frac{\partial \Lambda_k}{\partial k} = \frac{3G_k}{\pi}\,\frac{\Lambda_k\left(k^2 - \frac{2}{3}\Lambda_k\right)}{1 + \frac{3G_k}{2\pi}\left(k^2 - \frac{2}{3}\Lambda_k\right)} \quad ; \quad k\frac{\partial G_k}{\partial k} = \frac{3G_k^2}{\pi}\,\frac{k^2 - \frac{8}{3}\Lambda_k}{1 + \frac{3G_k}{2\pi}\left(k^2 - \frac{2}{3}\Lambda_k\right)}$$

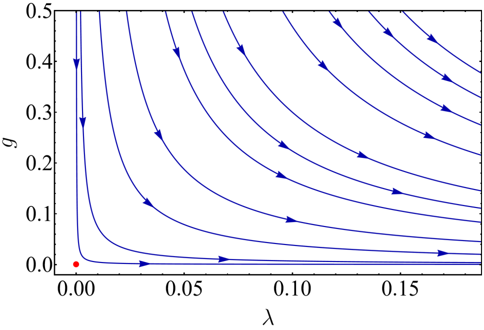
Dimless equations $t = \ln\frac{k}{k_0}$; $\lambda_t = \frac{\Lambda_k}{k^2}$; $g_t = k^2 G_k$

$$\frac{\partial \lambda_t}{\partial t} = -2\lambda_t + \frac{3g_t}{\pi}\,\frac{\lambda_t\left(1 - \frac{2}{3}\lambda_t\right)}{1 + \frac{3g_t}{2\pi}\left(1 - \frac{2}{3}\lambda_t\right)} \equiv \beta_\lambda(\lambda, g)$$

$$\frac{\partial g_t}{\partial t} = 2g_t + \frac{3g_t^2}{\pi}\,\frac{1 - \frac{8}{3}\lambda_t}{1 + \frac{3g_t}{2\pi}\left(1 - \frac{2}{3}\lambda_t\right)} \equiv \beta_g(\lambda, g)$$

Fixed points found from $\beta_\lambda = 0$; $\beta_g = 0$

RG flows and Fixed points



$(\lambda, g)_1 = (0, 0)$ axes $\lambda = 0$, $g = 0$ UV-repulsive/attractive respectively

$(\lambda, g)_2 = (0, -\pi/3)$ unphysical UV-attractive fixed point

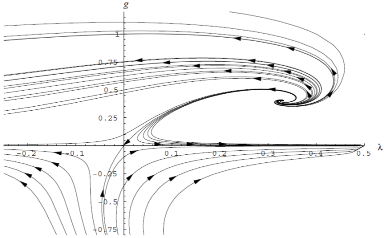
NO sign of any **physical UV-attractive** fixed point (**AS**)

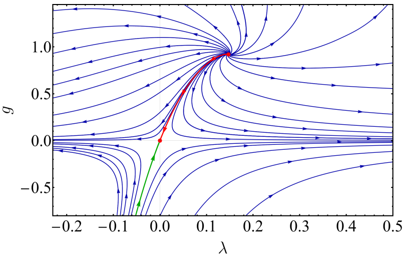
What generates the AS behaviour?

$$L\frac{\partial}{\partial L}S_L = 2\Lambda_L^2\, a^4 + 2\Lambda_L \left(L^2 - 8\right) a^2 + \frac{L^4}{3} - \frac{34L^2}{3} + \frac{1859}{45}$$

Identify the running scale k as $\boxed{k = L/a}$ (rather than $k = L/\bar{a}_L$)

$$k\frac{\partial}{\partial k}\Lambda_k = \frac{G_k}{\pi}\left[k^4 + 6\Lambda_k\left(k^2 + \Lambda_k\right) - \Lambda_k\frac{34k^2 + 48\,\Lambda_k}{6}\right] \quad ; \quad k\frac{\partial}{\partial k}G_k = -G_k^2\,\frac{34k^2 + 48\,\Lambda_k}{6\pi}$$





$(\lambda, g)_2 = (0.147, 0.918)$ UV-attractive fixed point (**AS**)

Is then the incorrect identification of k that gives rise to the AS flow?

Wetterich-Reuter equation

$\Gamma_k[g,\bar{g}]$ Effective Average Action ; $\kappa \equiv (32\pi G)^{-1/2}$; $\bar{g}_{\mu\nu}$ fixed background

$$k\,\partial_k \Gamma_k[g,\bar{g}] = \frac{1}{2}\,\mathrm{Tr}\left[\left(\kappa^{-2}\Gamma_k^{(2)}[g,\bar{g}] + R_k^{\mathrm{grav}}[\bar{g}]\right)^{-1} k\,\partial_k R_k^{\mathrm{grav}}[\bar{g}]\right] - \mathrm{Tr}\left[\left(-\mathcal{M}[g,\bar{g}] + R_k^{\mathrm{gh}}[\bar{g}]\right)^{-1} k\,\partial_k R_k^{\mathrm{gh}}[\bar{g}]\right]$$

M. Reuter, C. Wetterich

$\mathcal{M}[g,\bar{g}]$ classical kinetic term of the ghosts

$$\mathcal{M}[g,\bar{g}]^{\mu}{}_{\nu} = \bar{g}^{\mu\rho}\bar{g}^{\sigma\lambda}\bar{D}_\lambda\left(g_{\rho\nu}D_\sigma + g_{\sigma\nu}D_\rho\right) - \bar{g}^{\rho\sigma}\bar{g}^{\mu\lambda}\bar{D}_\lambda g_{\sigma\nu}D_\rho$$

$\bar{D}_\mu$ covariant derivative ; Christoffel symbols from $\bar{g}_{\mu\nu}$
Choice: $\bar{g}_{\mu\nu}$ of sphere radius a. Regulators $R_k^{\mathrm{grav}}[\bar{g}]$ and $R_k^{\mathrm{gh}}[\bar{g}]$ have the form

$$R_k[\bar{g}] = \mathcal{Z}_k k^2 R^{(0)}(-\Box/k^2) \quad , \quad \Box \equiv \bar{g}^{\mu\nu}\bar{D}_\mu\bar{D}_\nu$$

Modes w/ $p^2 > k^2$: integrated out ; modes w/ $p^2 < k^2$: suppressed by R_k:
$\Longrightarrow$ "Tr" effectively contains only eigenmodes of $-\Box$ with eigenvalues

$$p^2 \sim \frac{n^2}{a^2} \sim k^2$$

Eigenvalues of $-\Box$ are used as discriminant to introduce sliding scale k

Conclusions - 2

Pure gravity - RG

absence of running term k^4 in the RG equations when:

- identification of the physical running scale properly done

- truly diffeomorphism invariant measure (FV) used

$\Longrightarrow$ **No** Asymptotic Safety scenario

i.e. **No** UV-attractive Fixed point (as in QCD)

Gravity (Einstein-Hilbert) is an Effective Field Theory (only Gaussian fixed point as QED and ϕ^4)

Important: it is the **spurious** k^4 term (quartic div.) that **artificially** gives rise to the non-trivial UV-attractive FP of the AS scenario

Dulcis in fundo ... Let's move to ...

Scalar theory non minimally coupled to gravity

$$S = \frac{1}{16\pi G}\int \mathrm{d}x\,\sqrt{g}\,\left(-R + 2\Lambda_{cc}\right) + \int \mathrm{d}x\,\sqrt{g}\left[\frac{1}{2}\,g^{\mu\nu}\,\partial_\mu\phi\,\partial_\nu\phi + \frac{\xi}{2}R\phi^2 + V(\phi)\right]$$

Taking the metric $g^{(a)}_{\mu\nu}$ of a sphere of radius a

$$S^{(a)}[\phi] = \frac{\pi\Lambda_{cc}}{3G}\,a^4 - \frac{2\pi}{G}\,a^2 + \int \mathrm{d}x\,\sqrt{g^{(a)}}\left[\frac{1}{2}\,g^{(a)\,\mu\nu}\,\partial_\mu\phi\,\partial_\nu\phi + \frac{\xi}{2}\,\frac{12}{a^2}\,\phi^2 + V(\phi)\right]$$

One-loop correction δS^{1l} $\quad(\phi = \Phi + \eta \;\;;\;\;$ expand S up to $\eta^2)$

$$e^{-\delta S^{1l}} = \int \left[\mathcal{D}\mu(\eta)\right]\,e^{-S_2}$$

where $\left[\mathcal{D}\mu\right]$ Fradkin-Vilkovisky measure (arXiv:2506.05100)

$$S_2 \equiv \frac{1}{2}\int \mathrm{d}x\,\sqrt{g^{(a)}}\;\eta\left[-\Box_a + \frac{12\,\xi}{a^2} + V''(\Phi)\right]\eta \quad;\quad \left[\mathcal{D}\mu(\eta)\right] = \prod_x\left[\left(g^{(a)\,00}(x)\right)^{\frac{1}{2}}\left(g^{(a)}(x)\right)^{\frac{1}{4}}\,\mathrm{d}\eta(x)\right]$$

Dimensionless operators

$-\Box_a$ spin-0 Laplace-Beltrami operator for a sphere of radius a

$\left(g^{(a)\,00}(x)\right)^{1/2}\left(g^{(a)}(x)\right)^{1/4}$ comes from the integration over conjugate momenta (FV)

$g^{(a)}_{\mu\nu} = a^2\,\widetilde{g}_{\mu\nu}$ where the elements of $\widetilde{g}_{\mu\nu}$ are dimensionless and a-independent

$$\left(g^{(a)\,00}(x)\right)^{1/2}\left(g^{(a)}(x)\right)^{1/4} = a\left(\widetilde{g}^{\,00}(x)\right)^{1/2}\left(\widetilde{g}(x)\right)^{1/4}$$

Convenient redefinition: $\widehat{\eta} \equiv a\eta$

$$S_2 = \frac{1}{2}\int \mathrm{d}x\,\sqrt{\widetilde{g}}\;\widehat{\eta}\left[-\widetilde{\Box} + 12\,\xi + a^2\,V''(\Phi)\right]\widehat{\eta}$$

$-\widetilde{\Box}$ dimensionless spin-0 Laplace-Beltrami operator defined as $-\widetilde{\Box} \equiv -a^2\,\Box_a$

Since $\qquad \mathrm{d}\eta(x) = a^{-1}\mathrm{d}\widehat{\eta}(x)$

$\left[\mathcal{D}\mu(\eta)\right]$ can be written as $\qquad \left[\mathcal{D}\mu(\eta)\right] = \mathcal{A}\prod_x\left[\mathrm{d}\widehat{\eta}(x)\right]$

where a-independent terms such as $\prod_x\left(\widetilde{g}^{\,00}(x)\right)^{1/2}\left(\widetilde{g}(x)\right)^{1/4}$ are included in the factor $\mathcal{A}$

Dimensionless operators

$-\widetilde{\Box}$ dimensionless Laplacian $\Longrightarrow$ expand $\widehat{\eta}(x)$ with eigenfunctions $\phi_n^{(i)}(x)$ of $-\widetilde{\Box}$

$$\widehat{\eta} = \sum_{n,i} c_n^{(i)} \phi_n^{(i)}$$

Eigenvalues λ_n of $-\widetilde{\Box}$ and degeneracies D_n

$$\lambda_n = n^2 + 3n \qquad ; \qquad D_n = \frac{1}{3}\left(n+\frac{3}{2}\right)^3 - \frac{1}{12}\left(n+\frac{3}{2}\right)$$

Quantum correction to the action

$$e^{-\delta S} = \int \left[\mathcal{D}\mu(\eta)\right] e^{-S_2} = \widetilde{\mathcal{A}} \int \prod_{n,i} \mathrm{d}c_n^{(i)} e^{-\frac{1}{2}\sum_{n,i}\left[c_n^{(i)}\right]^2 \left(\lambda_n + 12\xi + a^2 V''(\Phi)\right)}$$

$$\Longrightarrow \qquad \Gamma = S^{(a)}[\Phi] + \frac{1}{2}\log\left[\det\left(-\widetilde{\Box} + 12\xi + a^2 V''(\Phi)\right)\right] + \mathcal{C}$$

Had we missed $\left(g^{(a)\,00}(x)\right)^{1/2}\left(g^{(a)}(x)\right)^{1/4} \Longrightarrow$ a-dependence of the determinant altered $\Longrightarrow$ determinant dimensionful $\Longrightarrow$ arbitrary scale μ needed to make argument of Log dimensionless

Calculation of $\log\det\left(-\widetilde{\Box} + 12\xi + a^2 V''(\Phi)\right)$

1 - Product of eigenvalues

Finite number N ($\gg 1$) of eigenvalues λ_n. Largest eigenvalue: $\lambda_N \sim N^2$ ($N-2$ for convenience)

$$\delta S^{1l} = \frac{1}{2} \sum_{n=0}^{N-2} \left[D_n \log \left(\lambda_n + 12\xi + a^2 V''(\Phi) \right) \right] + \mathcal{C}$$

$N =$ numerical UV cutoff

Expanding for $N \gg 1$

$$\delta S^{1l} = \frac{8\pi^2}{3} a^4 \left[- \frac{\left(V''(\Phi) \right)^2}{64\pi^2} \log N^2 + \frac{12}{a^2} \frac{V''(\Phi)}{384\pi^2} \left(N^2 + 2(1-6\xi) \log N^2 \right) \right] + \frac{N^4}{48} \left(-1 + 2 \log N^2 \right) - \frac{N^2}{72} \left(13 - 72\xi + 3 \log N^2 \right) + \left(2\xi(1-3\xi) - \frac{29}{180} \right) \log N^2 + \mathcal{C}$$

Consider Φ^4-theory : $V(\Phi) = \frac{m^2}{2}\Phi^2 + \frac{\lambda}{4!}\Phi^4$

One-loop effective action $\Gamma^{1l} = S^{(a)}[\Phi] + \delta S^{1l}$

$$\Gamma^{1l} = \frac{8\pi^2}{3} a^4 \Bigg\{ -\frac{1}{16\pi G}\left[1 - \frac{G\,m^2}{24\pi}\left(N^2 + 2(1-6\xi)\log N^2\right)\right]\frac{12}{a^2}$$
$$+ \frac{\Lambda_{cc}}{8\pi G}\left[1 - \frac{G\,m^4}{8\pi\Lambda_{cc}}\log N^2\right] + \frac{\xi}{2}\left[1 + \frac{\lambda}{384\pi^2\,\xi}\Big(N^2 + 2\,(1-6\xi)\log N^2\Big)\right]\frac{12}{a^2}\Phi^2$$
$$+ \frac{m^2}{2}\left[1 - \frac{\lambda}{32\pi^2}\log N^2\right]\Phi^2 + \frac{\lambda}{4!}\left[1 - \frac{3\lambda}{32\pi^2}\log N^2\right]\Phi^4 \Bigg\}$$
$$+ \frac{N^4}{48}\left(-1 + 2\log N^2\right) - \frac{N^2}{72}\left(13 - 72\xi + 3\log N^2\right) + \left(2\xi\,(1-3\xi) - \frac{29}{180}\right)\log N^2$$

2 - Proper time

Since $(-\widetilde{\Box} + 12\xi + a^2 V''(\Phi))$ dimensionless $\Longrightarrow$ regularize the determinant with dimensionless proper-time τ (lower cut $N \gg 1$). (λ_n and D_n eigenvalues and degeneracies)

$$\det(-\widetilde{\Box} + 12\xi + a^2 V''(\Phi)) = e^{-\int_{1/N^2}^{+\infty} \frac{d\tau}{\tau} \mathrm{K}(\tau)} \quad ; \quad \mathrm{K}(\tau) = \sum_{n=0}^{+\infty} D_n \, e^{-\tau\left(\lambda_n + 12\xi + a^2 V''(\Phi)\right)}$$

Perform first integration over τ and then sum over n with EML

$$\left[\sum_{n=n_i}^{n_f} f(n) = \int_{n_i}^{n_f} \mathrm{d}x\, f(x) + \frac{f(n_f)+f(n_i)}{2} + \sum_{k=1}^{p} \frac{B_{2k}}{(2k)!} \left(f^{(2k-1)}(n_f) - f^{(2k-1)}(n_i) \right) + R_{2p} \right.$$

p is an integer ; B_m are Bernoulli numbers ; R_{2p} is the rest

$$R_{2p} = \sum_{k=p+1}^{\infty} \frac{B_{2k}}{(2k)!} \left(f^{(2k-1)}(n_f) - f^{(2k-1)}(n_i) \right) = \frac{(-1)^{2p+1}}{(2p)!} \int_{n_i}^{n_f} \mathrm{d}x\, f^{(2p)}(x) B_{2p}(x - [x])$$

$B_n(x)$ Bernoulli polynomials ; $[x]$ integer part of x ; $f^{(i)}$ i-th derivative of f $\Big]$

Expanding the resulting expression of δS^{1l} for $N \gg 1$

$$\delta S^{1l} = \frac{8\pi^2}{3} a^4 \left[-\frac{\big(V''(\Phi)\big)^2}{64\pi^2} \log N^2 + \frac{12}{a^2}\frac{V''(\Phi)}{384\pi^2}\left(N^2 + 2\,(1-6\xi)\log N^2\right)\right]$$
$$-\frac{N^4}{24} - \frac{1-6\xi}{6} N^2 + \left(2\xi\,(1-3\xi) - \frac{29}{180}\right)\log N^2 + \mathcal{C}$$

Taking $V(\Phi) = \frac{m^2}{2}\Phi^2 + \frac{\lambda}{4!}\Phi^4 \quad \Longrightarrow \quad \Gamma^{1l}\ (= S^{(a)} + \delta S^{1l})$ is

$$\Gamma^{1l} = \frac{8\pi^2}{3} a^4 \Bigg\{ -\frac{1}{16\pi G}\left[1 - \frac{G\,m^2}{24\pi}\left(N^2 + 2(1-6\xi)\log N^2\right)\right]\frac{12}{a^2} + \frac{\Lambda_{cc}}{8\pi G}\left[1 - \frac{G\,m^4}{8\pi\Lambda_{cc}}\log N^2\right]$$
$$+\frac{\xi}{2}\left[1 + \frac{\lambda}{384\pi^2\,\xi}\left(N^2 + 2\,(1-6\xi)\log N^2\right)\right]\frac{12}{a^2}\Phi^2$$
$$+\frac{m^2}{2}\left[1 - \frac{\lambda}{32\pi^2}\log N^2\right]\Phi^2 + \frac{\lambda}{4!}\left[1 - \frac{3\lambda}{32\pi^2}\log N^2\right]\Phi^4 \Bigg\}$$
$$-\frac{N^4}{24} - \frac{1-6\xi}{6} N^2 + \left(2\xi\,(1-3\xi) - \frac{29}{180}\right)\log N^2$$

Apart from irrelevant a- and Φ-independent terms, the two results for Γ^{1l} coincide

One-loop corrections to $\frac{\Lambda_{cc}}{G}$, $\frac{1}{G}$, ξ, m^2 and λ

Comparing Γ^{1l} with $S^{(a)}[\Phi]$ we read the corrections to $\frac{\Lambda_{cc}}{G}$, $\frac{1}{G}$, ξ, m^2 and λ in terms of N

$$\frac{1}{G^{1l}} = \frac{1}{G}\left[1 - \frac{G\,m^2}{24\pi}\left(N^2 + 2(1-6\xi)\log N^2\right)\right]$$

$$\frac{\Lambda_{cc}^{1l}}{G^{1l}} = \frac{\Lambda_{cc}}{G}\left[1 - \frac{G\,m^4}{8\pi\Lambda_{cc}}\log N^2\right]$$

$$m^2_{1l} = m^2\left[1 - \frac{\lambda}{32\pi^2}\log N^2\right]$$

$$\lambda^{1l} = \lambda\left[1 - \frac{3\lambda}{32\pi^2}\log N^2\right]$$

$$\xi^{1l} = \xi\left[1 + \frac{\lambda}{384\pi^2\,\xi}\left(N^2 + 2\,(1-6\xi)\log N^2\right)\right]$$

Now connect N with Λ ($\sim M_P$ or string scale M_s) $\quad \Lambda = \frac{N}{a_{\rm dS}} = N\sqrt{\frac{\Lambda_{cc}}{3}} \quad \Longrightarrow$

For completeness we go back to the effective action and then move to the couplings

Effective Action

$$
\begin{aligned}
\Gamma^{1l} = \frac{8\pi^2}{3} a^4 \Bigg\{ & -\frac{1}{16\pi G}\left[1 - \frac{G\,m^2}{24\pi}\left(\frac{3\Lambda^2}{\Lambda_{cc}} + 2(1-6\xi)\log\frac{3\Lambda^2}{\Lambda_{cc}}\right)\right]\frac{12}{a^2} \\
& + \frac{\Lambda_{cc}}{8\pi G}\left[1 - \frac{G\,m^4}{8\pi\Lambda_{cc}}\log\frac{3\Lambda^2}{\Lambda_{cc}}\right] \\
& + \frac{\xi}{2}\left[1 + \frac{\lambda}{384\pi^2\,\xi}\left(\frac{3\Lambda^2}{\Lambda_{cc}} + 2\,(1-6\xi)\log\frac{3\Lambda^2}{\Lambda_{cc}}\right)\right]\frac{12}{a^2}\Phi^2 \\
& + \frac{m^2}{2}\left[1 - \frac{\lambda}{32\pi^2}\log\frac{3\Lambda^2}{\Lambda_{cc}}\right]\Phi^2 + \frac{\lambda}{4!}\left[1 - \frac{3\lambda}{32\pi^2}\log\frac{3\Lambda^2}{\Lambda_{cc}}\right]\Phi^4 \Bigg\} \\
& - \frac{3\Lambda^4}{8\Lambda_{cc}^2} - \frac{1-6\xi}{2}\frac{\Lambda^2}{\Lambda_{cc}} + \left(2\xi\,(1-3\xi) - \frac{29}{180}\right)\log\frac{3\Lambda^2}{\Lambda_{cc}}
\end{aligned}
$$

... Let's read how the couplings get modified ...

$$\frac{\Lambda_{cc}^{1l}}{8\pi G^{1l}} = \frac{\Lambda_{cc}}{8\pi G}\left[1 - \frac{G\,m^4}{8\pi\Lambda_{cc}}\log\frac{3\Lambda^2}{\Lambda_{cc}}\right] \quad ; \quad \frac{1}{G^{1l}} = \frac{1}{G}\left[1 - \frac{G\,m^2}{24\pi}\left(\frac{3\Lambda^2}{\Lambda_{cc}} + 2(1-6\xi)\log\frac{3\Lambda^2}{\Lambda_{cc}}\right)\right]$$

$$m_{1l}^2 = m^2\left[1 - \frac{\lambda}{32\pi^2}\log\frac{3\Lambda^2}{\Lambda_{cc}}\right] \quad ; \quad \lambda^{1l} = \lambda\left[1 - \frac{3\lambda}{32\pi^2}\log\frac{3\Lambda^2}{\Lambda_{cc}}\right]$$

$$\xi^{1l} = \xi\left[1 + \frac{\lambda}{384\pi^2\,\xi}\left(\frac{3\Lambda^2}{\Lambda_{cc}} + 2\,(1-6\xi)\log\frac{3\Lambda^2}{\Lambda_{cc}}\right)\right]$$

Quartic self-coupling - only mild logarithmic correction (coincides with flat space-time result)

Scalar mass $\quad \delta m^2 \sim \log\Lambda$ rather than $\sim \Lambda^2$: **no quadratic divergence**

Usual result $\quad \delta m^2 \sim \Lambda^2$: enormus fine-tuning

Present result $\quad$ we may well have $m^2(\Lambda) \ll \Lambda^2$

No Naturalness Problem for the scalar mass??

If SM embedded in SUSY, GUT, ..., fields of heavy mass M coupled to Higgs $\Rightarrow \delta m^2 \propto M^2$.
Physical mechanism that disposes of these contributions and makes $m_H^2 \sim (125\,\text{GeV})^2$ needed !!

... work in progress ...

Still this is an important physical result, obtained within the Wilsonian framework, where **physical cutoff built-in**: Absence of quadratic divergence not due to "technical tricks" (dimensional, zeta function regularization, ...) nor to physical cancellations (SUSY, ...)

... still reading the couplings ...

$$\frac{\Lambda_{cc}^{1l}}{G^{1l}} = \frac{\Lambda_{cc}}{G}\left[1 - \frac{G\,m^4}{8\pi\Lambda_{cc}} \log\frac{3\Lambda^2}{\Lambda_{cc}}\right] \quad ; \quad \frac{1}{G^{1l}} = \frac{1}{G}\left[1 - \frac{G\,m^2}{24\pi}\left(\frac{3\Lambda^2}{\Lambda_{cc}} + 2(1-6\xi)\log\frac{3\Lambda^2}{\Lambda_{cc}}\right)\right]$$

$$m_{1l}^2 = m^2\left[1 - \frac{\lambda}{32\pi^2}\log\frac{3\Lambda^2}{\Lambda_{cc}}\right] \quad ; \quad \lambda^{1l} = \lambda\left[1 - \frac{3\lambda}{32\pi^2}\log\frac{3\Lambda^2}{\Lambda_{cc}}\right]$$

$$\xi^{1l} = \xi\left[1 + \frac{\lambda}{384\pi^2\,\xi}\left(\frac{3\Lambda^2}{\Lambda_{cc}} + 2\,(1-6\xi)\log\frac{3\Lambda^2}{\Lambda_{cc}}\right)\right]$$

Non-minimal coupling ξ : besides a mild logarithmic correction (present in previous literature)

ξ receives a quadratically divergent contribution

UV sensitivity of m^2 and ξ inverted : $m^2 \sim \log\Lambda \quad ; \quad \xi \sim \Lambda^2$.

Phenomenological remarks

Higgs boson mass confronted with measured $m_{\rm H}^2 \sim (125\,{\rm GeV})^2 \Longrightarrow$ quadratic sensitivity to Λ gives rise to severe naturalness problem. Much less is known on the experimental value of ξ. Correction $\delta\xi \sim \Lambda^2$ can be easily handled, and does not seem to be worrisome

... still reading the couplings ...

$$\frac{\Lambda_{cc}^{1l}}{G^{1l}} = \frac{\Lambda_{cc}}{G}\left[1 - \frac{G\,m^4}{8\pi\Lambda_{cc}}\log\frac{3\Lambda^2}{\Lambda_{cc}}\right] \quad ; \quad \frac{1}{G^{1l}} = \frac{1}{G}\left[1 - \frac{G\,m^2}{24\pi}\left(\frac{3\Lambda^2}{\Lambda_{cc}} + 2(1-6\xi)\log\frac{3\Lambda^2}{\Lambda_{cc}}\right)\right]$$

$$m_{1l}^2 = m^2\left[1 - \frac{\lambda}{32\pi^2}\log\frac{3\Lambda^2}{\Lambda_{cc}}\right] \quad ; \quad \lambda^{1l} = \lambda\left[1 - \frac{3\lambda}{32\pi^2}\log\frac{3\Lambda^2}{\Lambda_{cc}}\right]$$

$$\xi^{1l} = \xi\left[1 + \frac{\lambda}{384\pi^2\,\xi}\left(\frac{3\Lambda^2}{\Lambda_{cc}} + 2\,(1-6\xi)\log\frac{3\Lambda^2}{\Lambda_{cc}}\right)\right]$$

Vacuum energy $\rho_{\rm vac} = \frac{\Lambda_{cc}}{8\pi G}$. Radiative correction $\sim \log\Lambda$: *no quartic/quadratic divergence*

Note : $\log\Lambda$ correction multiplied by $m^4 \Longrightarrow$ for SM masses $m \sim \mu_F$ still left with (at least) 50 orders of magnitude discrepancy with measured vacuum energy. Absence of Λ^4 and Λ^2 corrections sheds some light on the CC problem, and makes it less severe

Inverse Newton constant - Quadratically UV-sensitive contribution as usual

Comparison with the literature

Calculation usually performed within the heat kernel expansion. Quadratically divergent contribution to the mass typically found (obviously dimensional regularization excluded from these considerations: trick to cancel powerlike divergences)

Let us go back to Γ^{1l} : temporarily connect N and Λ through

$$\Lambda = \frac{N}{a} \quad \text{rather than} \quad \Lambda = \frac{N}{a_{\text{dS}}}$$

$$\begin{aligned}
\Gamma^{1l} = \frac{8\pi^2}{3} a^4 \Bigg\{ & -\frac{1}{16\pi G}\left[1 + \frac{1-6\xi}{12\pi} G\Lambda^2 - \frac{G\, m^2}{24\pi}\left(2(1-6\xi)\log\left(a^2\Lambda^2\right)\right)\right]\frac{12}{a^2} \\
& + \frac{\Lambda_{\text{cc}}}{8\pi G}\left[1 - \frac{G}{8\pi\Lambda_{\text{cc}}}\Lambda^4 + \frac{m^2 G}{4\pi\Lambda_{\text{cc}}}\Lambda^2 - \frac{G\, m^4}{8\pi\Lambda_{\text{cc}}}\log\left(a^2\Lambda^2\right)\right] \\
& + \frac{\xi}{2}\left[1 + \frac{\lambda}{384\pi^2\,\xi}\left(2\,(1-6\xi)\log\left(a^2\Lambda^2\right)\right)\right]\frac{12}{a^2}\Phi^2 \\
& + \frac{m^2}{2}\left[1 + \frac{\lambda\,\Lambda^2}{32\pi^2\, m^2} - \frac{\lambda}{32\pi^2}\log\left(a^2\Lambda^2\right)\right]\Phi^2 + \frac{\lambda}{4!}\left[1 - \frac{3\lambda}{32\pi^2}\log\left(a^2\Lambda^2\right)\right]\Phi^4 \Bigg\} \\
& + \left(2\xi\,(1-3\xi) - \frac{29}{180}\right)\log\left(a^2\Lambda^2\right)
\end{aligned}$$

From $\frac{N}{a_{dS}}$ to $\frac{N}{a}$

$$\Gamma^{1l} = \frac{8\pi^2}{3} a^4 \Bigg\{ -\left[1 - \frac{G\, m^2}{24\pi}\left(N^2 + 2(1-6\xi)\log N^2\right)\right]\frac{1}{16\pi G}\frac{12}{a^2} + \left[1 - \frac{G\, m^4}{8\pi\Lambda_{cc}}\log N^2\right]\frac{\Lambda_{cc}}{8\pi G}$$
$$+ \left[1 + \frac{\lambda}{384\pi^2\,\xi}\left(N^2 + 2\,(1-6\xi)\log N^2\right)\right]\frac{\xi}{2}\frac{12}{a^2}\Phi^2 + \left[1 - \frac{\lambda}{32\pi^2}\log N^2\right]\frac{m^2}{2}\Phi^2 + \left[1 - \frac{3\lambda}{32\pi^2}\log N^2\right]\frac{\lambda}{4!}\Phi^4 \Bigg\}$$
$$-\frac{N^4}{24} - \frac{1-6\xi}{6} N^2 + \left(2\xi\,(1-3\xi) - \frac{29}{180}\right)\log N^2$$

Legenda: **Green** $\rightarrow \frac{\Lambda_{cc}}{8\pi G}$; **Red** $\rightarrow m^2$; **Blue** $\rightarrow \frac{1}{16\pi G}$

$$\Gamma^{1l} = \frac{8\pi^2}{3} a^4 \Bigg\{ -\left[1 + \frac{1-6\xi}{12\pi} G\Lambda^2 - \frac{G\, m^2}{24\pi}\left(2(1-6\xi)\log\left(a^2\Lambda^2\right)\right)\right]\frac{1}{16\pi G}\frac{12}{a^2}$$
$$+ \left[1 - \frac{G}{8\pi\Lambda_{cc}}\Lambda^4 + \frac{m^2 G}{4\pi\Lambda_{cc}}\Lambda^2 - \frac{G\, m^4}{8\pi\Lambda_{cc}}\log\left(a^2\Lambda^2\right)\right]\frac{\Lambda_{cc}}{8\pi G}$$
$$+ \left[1 + \frac{\lambda}{384\pi^2\,\xi}\left(2\,(1-6\xi)\log\left(a^2\Lambda^2\right)\right)\right]\frac{\xi}{2}\frac{12}{a^2}\Phi^2$$
$$+ \left[1 + \frac{\lambda\,\Lambda^2}{32\pi^2\, m^2} - \frac{\lambda}{32\pi^2}\log\left(a^2\Lambda^2\right)\right]\frac{m^2}{2}\Phi^2 + \left[1 - \frac{3\lambda}{32\pi^2}\log\left(a^2\Lambda^2\right)\right]\frac{\lambda}{4!}\Phi^4 \Bigg\}$$
$$+ \left(2\xi\,(1-3\xi) - \frac{29}{180}\right)\log\left(a^2\Lambda^2\right)$$

Speculations (Conclusions ?)

Curved vs. flat spacetime

Quantum fluctuations usually computed directly in flat spacetime … However …

Evidence of positive vacuum energy $\frac{\Lambda_{cc}}{8\pi G} > 0$: Flat spacetime not suitable cosmological description

Quantum corrections should be computed on curved background

Flat spacetime as limit … different from flat ab initio …

Our calculations indicate that usual methods may fail …

Quadratic divergence in ξ instead of m^2 cannot be detected in flat spacetime computations, since in this case $R\phi^2 = 0$

BACK-UP SLIDES

Calculation with EML in proper time regularization

$(-\widetilde{\Box}^{(s)} - \alpha)$ dimensionless $\Longrightarrow$ determinants regularized in terms of a **dimensionless proper-time** τ (lower cut: **number** $N \gg 1$)

$$\det{}_i(-\widetilde{\Box}^{(s)} - \alpha) = e^{-\int_{1/N^2}^{+\infty} \frac{d\tau}{\tau} \mathrm{K}_i^{(s)}(\tau)} \quad ; \quad \mathrm{K}_i^{(s)}(\tau) = \sum_{n=s+i}^{+\infty} D_n^{(s)}\, e^{-\tau\left(\lambda_n^{(s)} - \alpha\right)}$$

After integration over τ, sum over n performed with EML sum formula

$$\sum_{n=n_i}^{n_f} f(n) = \int_{n_i}^{n_f} dx\, f(x) + \frac{f(n_f) + f(n_i)}{2} + \sum_{k=1}^{p} \frac{B_{2k}}{(2k)!}\left(f^{(2k-1)}(n_f) - f^{(2k-1)}(n_i)\right) + R_{2p}$$

p is an integer, B_m are Bernoulli numbers, R_{2p} is the rest given by

$$R_{2p} = \sum_{k=p+1}^{\infty} \frac{B_{2k}}{(2k)!}\left(f^{(2k-1)}(n_f) - f^{(2k-1)}(n_i)\right) = \frac{(-1)^{2p+1}}{(2p)!} \int_{n_i}^{n_f} dx\, f^{(2p)}(x) B_{2p}(x - [x])$$

$B_n(x)$ are the Bernoulli polynomials, $[x]$ the integer part of x, and $f^{(i)}$ the i-th derivative of f with respect to its argument

Proper time RG

Wilsonian RG strategy implemented introducing an IR regulator k in the one-loop result

$$\int_{1/\Lambda_{\rm cut}^2}^{+\infty} {\rm d}s \quad \longrightarrow \quad \int_{1/\Lambda_{\rm cut}^2}^{1/k^2} {\rm d}s \ ,$$

taking the derivative with respect to k, and finally realizing the RG improvement of the one-loop result

Equivalently introducing a smooth function $f_k(s)$ that interpolate between $f_k(s) \approx 0$ for $s \gg k^{-2}$ and $f_k(s) \approx 1$ for $s \ll k^{-2}$ $\quad\Rightarrow\quad$ RG equation for the action

$$\partial_t \widehat{S}_k[g,\bar{g}] = -\frac{1}{2}\,{\rm Tr} \int_0^{+\infty} \frac{{\rm d}s}{s}\,\partial_t f_k(s)\left[e^{-s\,\widehat{S}_k^{(2)}} - 2\,e^{-s\,S_{\rm ghost}^{(2)}}\right] .$$

where $\widehat{S}\,[\bar{h};g] \equiv S[\bar{g}+\bar{h}] \, + \, S_{\rm gf}[\bar{h};\bar{g}]$

Background metric $\bar{g}_{\mu\nu} = g^{(a)}_{\mu\nu}$, Einstein-Hilbert truncation for $\widehat{S}_k$ $\Rightarrow$ $\widehat{S}_k^{(2)}$ contains dimensionful Laplace-Beltrami operators $-\Box$ for the sphere of radius a (and different spins $0,1,2$) whose eigenvalues $\widehat{\lambda}_n$ go like $\widehat{\lambda}_n \sim \frac{n^2}{a^2}$

The term $\partial_t f_k(s)$ effectively selects the eigenmodes of $-\Box$ whose corresponding eigenvalues lie in a narrow range ("infinitesimal shell") around k^2, i.e. $\widehat{\lambda}_n \sim k^2$

As for the effective average action formalism, here the running scale k is identified through the relation $k = L/a$, and the **same conclusions on the UV-attractive fixed point of the asymptotic safety scenario hold true**

Expansion of $\widehat{h}_{\mu\nu}$, v^*_ρ and v_σ

We indicate with $h_n^{\mu\nu(i)}$ (transverse-traceless), $\xi_n^{\mu(i)}$ (transverse) and $\phi_n^{(i)}$ the pure spin-2, spin-1 and spin-0 eigenfunctions of the Laplace-Beltrami operator on the sphere of unitary radius that are normalized as

$$\delta^{ij}\delta_{nm} = \int \mathrm{d}^4x\,\sqrt{\widetilde{g}}\,h_n^{\mu\nu(i)}(x)h_{\mu\nu}^{m(j)}(x) = \int \mathrm{d}^4x\,\sqrt{\widetilde{g}}\,\xi_n^{\mu(i)}(x)\xi_\mu^{m(j)}(x) = \int \mathrm{d}^4x\,\sqrt{\widetilde{g}}\;\phi_n^{(i)}(x)\,\phi_m^{(j)}(x)\,,$$

corresponding to the eigenvalues $\lambda_n^{(2)}$, $\lambda_n^{(1)}$ and $\lambda_n^{(0)}$ respectively. The modes $\{h_n^{\mu\nu},\, v_n^{\mu\nu},\, w_n^{\mu\nu},\, z_n^{\mu\nu}\}$, with

$$\begin{aligned}
v_n^{\mu\nu} &= \left[\frac{1}{2}\left(\lambda_n^{(1)}-3\right)\right]^{-\frac{1}{2}}\nabla^{(\mu}\xi_n^{\nu)}\,, \quad n=2,\ldots\,,\\
w_n^{\mu\nu} &= \left[\lambda_n^{(0)}\left(\frac{3}{4}\lambda_n^{(0)}-3\right)\right]^{-\frac{1}{2}}\left(\nabla^\mu\nabla^\nu-\frac{1}{4}\widetilde{g}^{\mu\nu}\Box\right)\phi_n\,, \quad n=2,\ldots\,,\\
z_n^{\mu\nu} &= \frac{1}{2}\widetilde{g}^{\mu\nu}\phi_n\,, \quad n=0,1,2,\ldots\,,
\end{aligned}$$

of which we do not write explicitly the degeneracy indexes form the orthonormal basis for symmetric tensors.

Moreover, defining the longitudinal vector modes

$$l_n^\mu = \left(\lambda_n^{(0)}\right)^{-\frac{1}{2}} \nabla^\mu \phi_n\,, \quad n = 1, 2, \ldots\,,$$

the latter, together with the transverse modes ξ_n^μ, form the orthonormal basis for vectors.

Expand the graviton field $\widehat{h}^{\mu\nu}$ as

Taylor, Veneziano

$$\widehat{h}^{\mu\nu} = \sum_{n=2}^{\infty} a_n h_n^{\mu\nu} + \sum_{n=2}^{\infty} b_n v_n^{\mu\nu} + \sum_{n=2}^{\infty} c_n w_n^{\mu\nu} + \sum_{n=0}^{\infty} e_n z_n^{\mu\nu}$$

$$\widehat{h} \equiv \widetilde{g}_{\mu\nu} \widehat{h}^{\mu\nu} = 2 \sum_{n=0}^{\infty} e_n \phi_n\,,$$

and the ghost field v^μ as

$$v^\mu = \sum_{n=1}^{\infty} g_n\, \xi_n^\mu + \sum_{n=1}^{\infty} f_n\, l_n^\mu$$

so that we have

$$
\begin{aligned}
64\pi G\left(S_2+S_{\rm gf}\right) = & \sum_{n=2}^{\infty} a_n^2\left[\lambda_n^{(2)}-2a^2\Lambda+8\right] \\
& +\sum_{n=2}^{\infty} b_n^2\left[\xi^{-1}\left(\lambda_n^{(1)}-3\right)-2a^2\Lambda+6\right] \\
& +\sum_{n=2}^{\infty} c_n^2\left[\xi^{-1}\left(\frac{3}{4}\lambda_n^{(0)}-6\right)-\frac{\lambda_n^{(0)}}{2}-2a^2\Lambda+6\right] \\
& +\sum_{n=0}^{\infty} e_n^2\left[\frac{-3+\xi^{-1}}{2}\lambda_n^{(0)}+2a^2\Lambda\right] \\
& +\sum_{n=2}^{\infty} 2e_n c_n(\xi^{-1}-1)\left[\lambda_n^{(0)}\left(\frac{3}{4}\lambda_n^{(0)}-3\right)\right]^{\frac{1}{2}}
\end{aligned}
$$

$$
32\pi G\, S_{\rm ghost} = \sum_{n=1}^{\infty} g_n^*\, g_n\left(\lambda_n^{(1)}-3\right)+\sum_{n=1}^{\infty} f_n^*\, f_n\left(\lambda_n^{(0)}-6\right)\ .
$$

Therefore, the functional measure can be written as (defined as)

$$\mathcal{D}\widehat{h}_{\mu\nu}\,\mathcal{D}v^*_\rho\,\mathcal{D}v_\sigma \equiv \frac{1}{V_{SO(5)}}\prod_{n=2}^{\infty}\mathrm{d}a_n\prod_{n=2}^{\infty}\mathrm{d}b_n\prod_{n=2}^{\infty}\mathrm{d}c_n\prod_{n=0}^{\infty}\mathrm{d}e_n\prod_{n=2}^{\infty}\mathrm{d}g^*_n\prod_{n=2}^{\infty}\mathrm{d}g_n\prod_{n=1}^{\infty}\mathrm{d}f^*_n\prod_{n=1}^{\infty}\mathrm{d}f_n\,,$$

Notice that there is no integration over the zero modes g_1^* and g_1 of S_{ghost}. The corresponding ghost fields are proportional to the ten Killing vectors ξ_1^μ. These zero eigenmodes correspond to residual gauge invariances which are not eliminated by gauge fixing in the presence of an $SO(5)$ spherical symmetry. Overcounting has been compensated by inserting the explicit group-volume factor $V_{SO(5)}$

Additional comments

The terms $\frac{1}{2}\log\left(2a^2\Lambda_{\rm cc}\right)$, $\mathcal{B}$ and $\mathcal{F}(a^2\Lambda_{\rm cc})$ (the finite term of the sum over the eigenvalues) are negligible $\mathcal{O}(1)$ contributions to $\delta S^{1l}_{\rm grav}$

The constant terms (proportional to a^0) in principle could be interpreted as corrections to $\int \mathrm{d}^4x\,\sqrt{g}\,R^2$ rather then as constants to be discarded

... Due to the high symmetry of the background considered (sphere), it is impossible to distinguish between constant terms and corrections to R^2

... since our universe seems to be well described by the Einstein-Hilbert action (with cosmological constant) even at large energy scales, we rather expect these terms to be interpreted as inessential constants ...

This question should be further investigated ...

Finite term of the sum over the eigenvalues

$$
\begin{aligned}
\mathcal{F}(a^2\Lambda) = {} & 9\Lambda a^2 - \frac{1}{6}\Lambda\sqrt{8\Lambda a^2+9}\log\Gamma\left(\frac{7}{2}-\frac{1}{2}\sqrt{8\Lambda a^2+9}\right)a^2 - 5\Lambda\psi^{(-2)}\left(\frac{1}{2}\left(\sqrt{8a^2\Lambda-15}+7\right)\right)a^2 \\
& - 5\Lambda\psi^{(-2)}\left(\frac{7}{2}-\frac{1}{2}\sqrt{8a^2\Lambda-15}\right)a^2 - \Lambda\psi^{(-2)}\left(\frac{1}{2}\left(\sqrt{8\Lambda a^2+9}+7\right)\right)a^2 \\
& - \Lambda\psi^{(-2)}\left(\frac{7}{2}-\frac{1}{2}\sqrt{8\Lambda a^2+9}\right)a^2 + \frac{1}{6}\Lambda\log\Gamma\left(\frac{1}{2}\left(\sqrt{8\Lambda a^2+9}+7\right)\right)\sqrt{8\Lambda a^2+9}\,a^2 \\
& - 5\log(120) + \frac{49\log(A)}{3} - 2\sqrt{\frac{11}{3}}\log\Gamma\left(\frac{1}{2}\left(\sqrt{33}+7\right)\right) \\
& - \frac{5}{6}\left(a^2\Lambda-5\right)\sqrt{8a^2\Lambda-15}\log\Gamma\left(\frac{7}{2}-\frac{1}{2}\sqrt{8a^2\Lambda-15}\right) \\
& - \frac{1}{6}\sqrt{8\Lambda a^2+9}\log\Gamma\left(\frac{7}{2}-\frac{1}{2}\sqrt{8\Lambda a^2+9}\right) + 3\psi^{(-4)}(1) + 3\psi^{(-4)}(6) + \psi^{(-4)}\left(\frac{7}{2}-\frac{\sqrt{33}}{2}\right) \\
& + \psi^{(-4)}\left(\frac{1}{2}\left(\sqrt{33}+7\right)\right) - 5\psi^{(-4)}\left(\frac{1}{2}\left(\sqrt{8a^2\Lambda-15}+7\right)\right) - 5\psi^{(-4)}\left(\frac{7}{2}-\frac{1}{2}\sqrt{8a^2\Lambda-15}\right) \\
& - \psi^{(-4)}\left(\frac{1}{2}\left(\sqrt{8\Lambda a^2+9}+7\right)\right) - \psi^{(-4)}\left(\frac{7}{2}-\frac{1}{2}\sqrt{8\Lambda a^2+9}\right) + \frac{15\psi^{(-3)}(1)}{2} - \frac{15\psi^{(-3)}(6)}{2} \\
& - \frac{1}{2}\sqrt{33}\psi^{(-3)}\left(\frac{1}{2}\left(\sqrt{33}+7\right)\right) - \frac{5}{2}\sqrt{8a^2\Lambda-15}\,\psi^{(-3)}\left(\frac{7}{2}-\frac{1}{2}\sqrt{8a^2\Lambda-15}\right)
\end{aligned}
$$