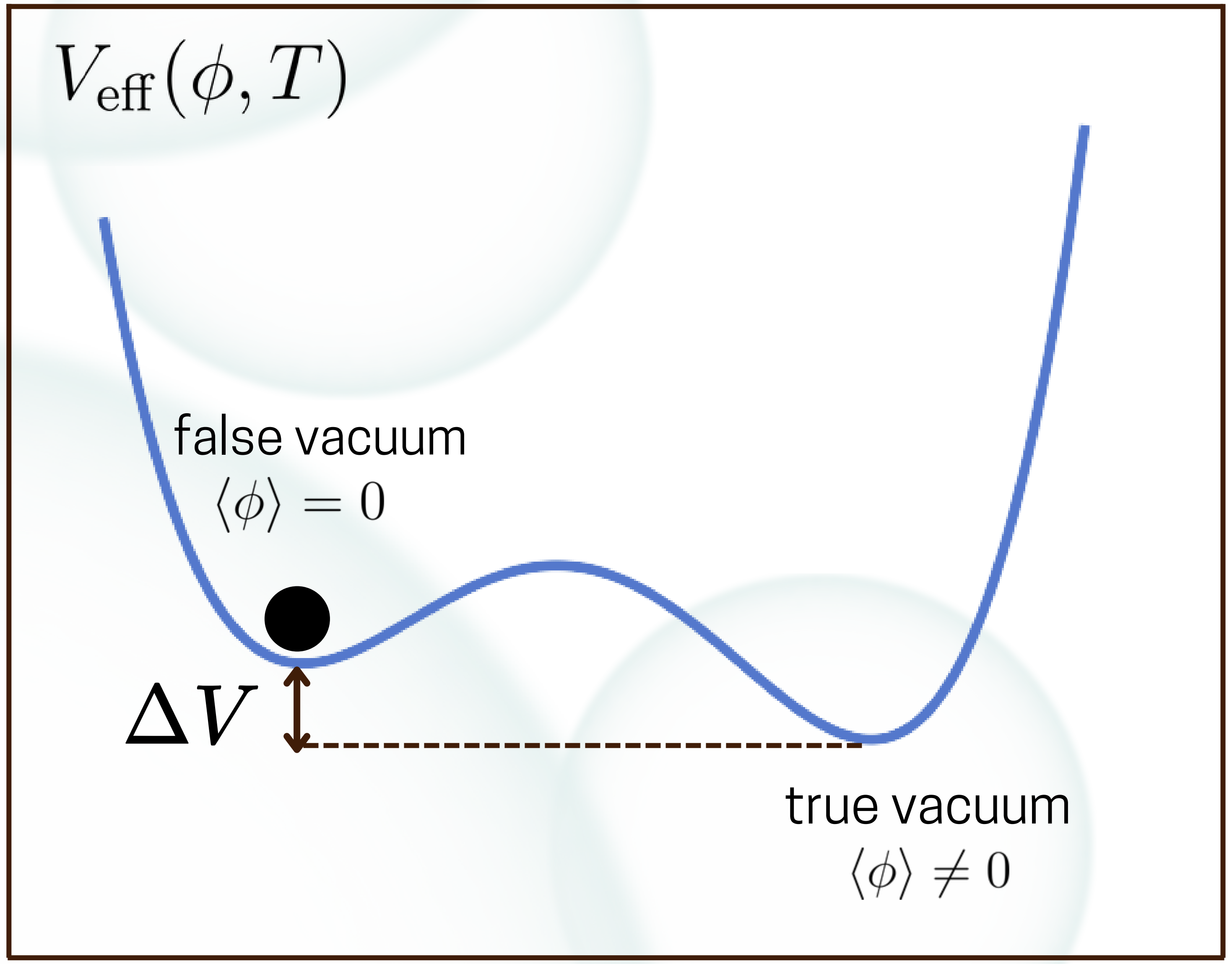
The Equation of State of the Universe after a First-Order Phase Transition

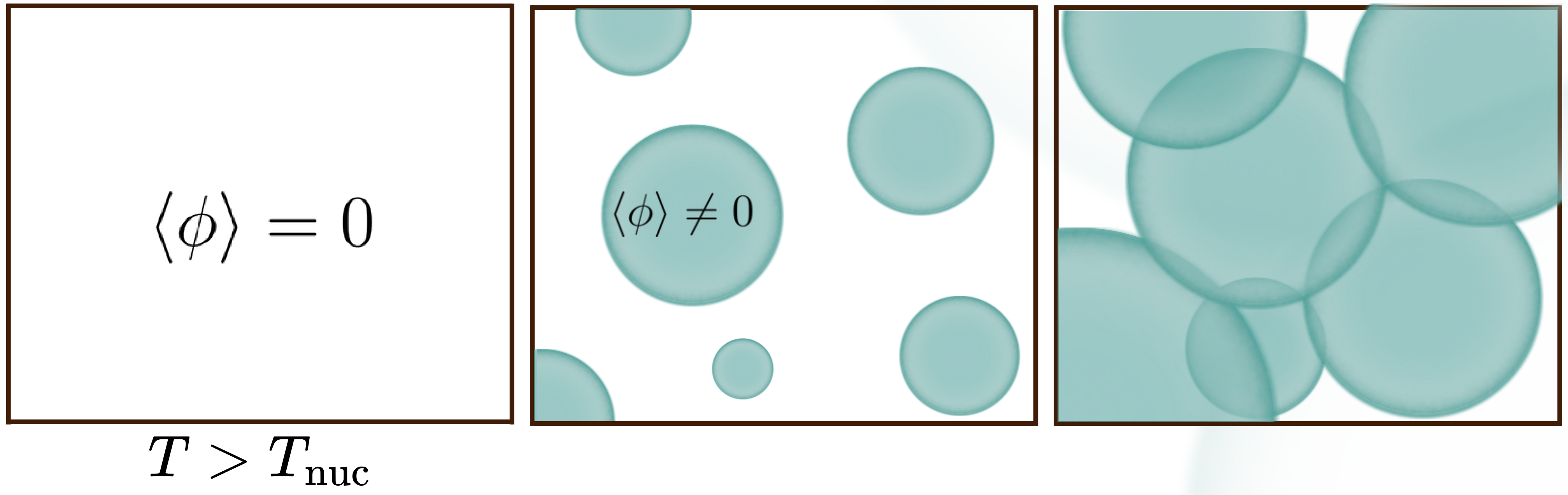
Henda Mansour

In collaboration with:
Yann Gouttenoire and Felix Kahlhoefer

First-Order Phase Transitions

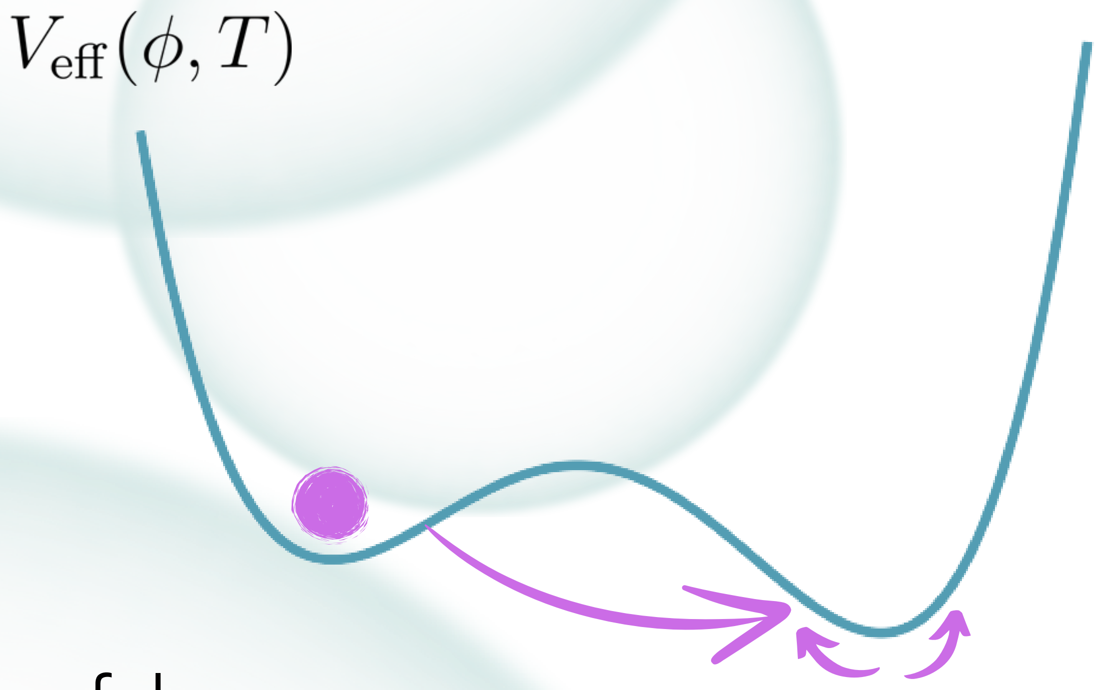


The transition proceeds through bubble nucleation:



+ The scalar field acts like a cosmological constant before the transition: $\omega_\phi = -1$
+ For strong supercooling $\Delta V > \rho_{\text{rad}}(T_{\text{PT}})$
+ If the scalar field decays slowly, there will be a period of **scalar field domination** after percolation

Scalar field domination after FOPT



false vacuum

$$\langle \phi \rangle = 0$$

true vacuum

$$\langle \phi \rangle \neq 0$$

If reheating is slow, such that we can focus only on the scalar dynamics:
After the transition, the field oscillates around the new minimum

→ Matter domination?

Equation of state: $\langle \omega \rangle = \dfrac{\langle p \rangle}{\langle \rho \rangle}$ ($\langle p \rangle$ → pressure, $\langle \rho \rangle$ → energy density)

In the case of coherent oscillations:

Around the minimum:

$$\langle \omega \rangle = \frac{\langle p_\phi \rangle}{\langle \rho_\phi \rangle} = \frac{\frac{1}{2}\langle \dot{\phi}^2 \rangle - \langle V(\phi) \rangle}{\frac{1}{2}\langle \dot{\phi}^2 \rangle + \langle V(\phi) \rangle}$$

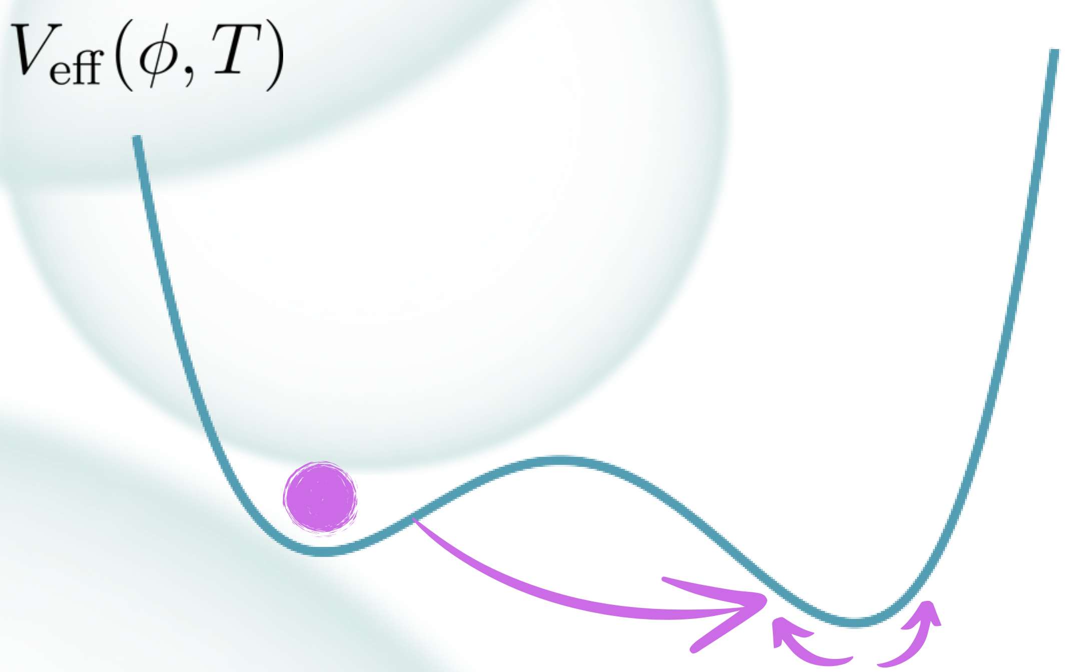
Virial Theorem

$V(\phi) \sim \phi^k \quad \Rightarrow \quad \langle E_{\text{kin}} \rangle = \frac{k}{2}\langle V(\phi) \rangle \quad \Rightarrow \quad$ For: $k = 2 \quad \omega = 0$

$$\Rightarrow \quad \omega = \frac{k-2}{k+2}$$

... It is however not as simple for a scalar field configuration post collision

Scalar field domination after FOPT

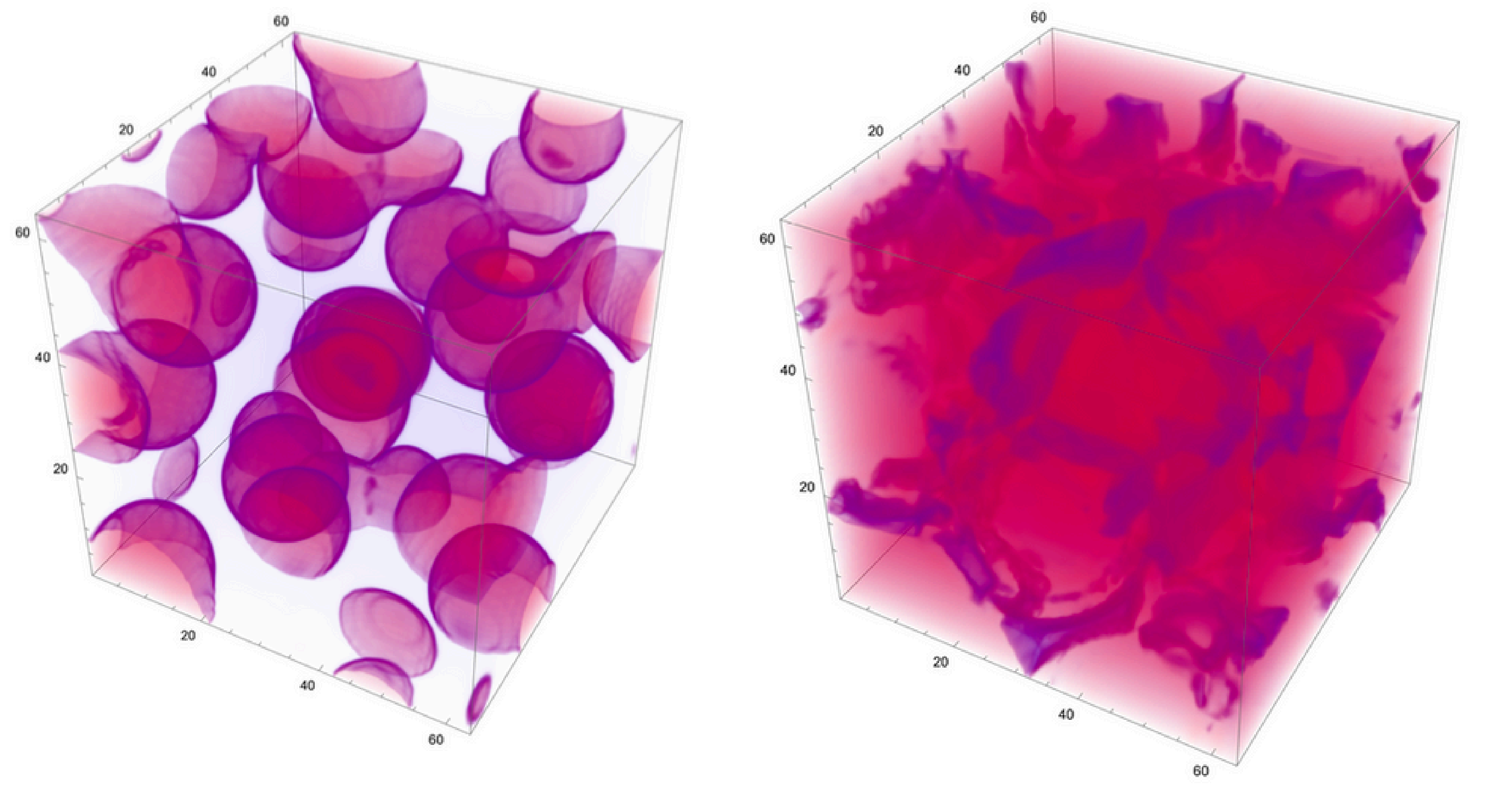


1. Non-vanishing gradients have to be taken into consideration
2. Bubble walls are relativistic:

→ relativistic scalar waves?

→ radiation domination?

Bubble nucleation and percolation on the lattice:



Spatially inhomogeneous

$$p_\phi = \frac{1}{2}\dot{\phi}^2 - \frac{1}{6}(\nabla\phi)^2 - V(\phi)$$

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla\phi)^2 + V(\phi)$$

Why does it matter ?

Particle production :

Dilution of pre-existing abundances in the case of matter domination (MD)

(Co. et al. (2015), Cirelli et al. (2016), Bishara et al. (2024) and many others)

Production of primordial black holes :

Smaller overdensity threshhold for collapse during MD

(PBH production during matter domination (Harada et al. arXiv: 1609.01588), PBH production during FOPTs: Y. Gouttenoire and T. Volansky, arXiv: 2305.04942)

Gravitational Waves: Could impact the spectrum of GW expected

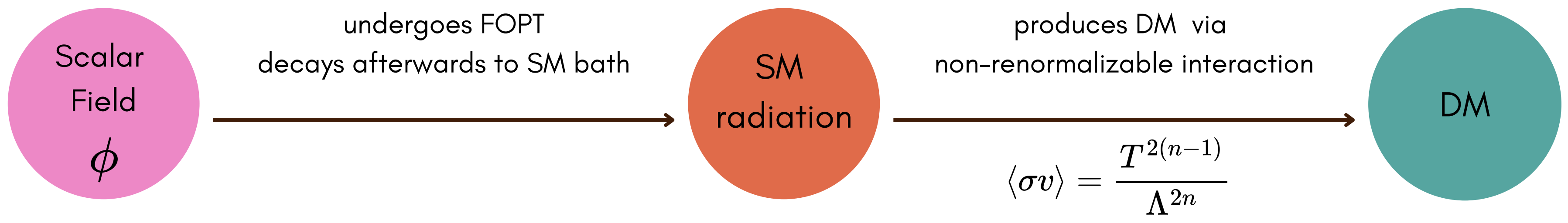
General impact of MD : (Kazunori Nakayama et al. (2008), Boyle and Steinhardt (2008), Seto and Yokoyama (2003), D'Eramo and Schmitz (2019)....)

More specific to FOPTs: (Ellis et al. (2019) and (2020))

Why does it matter ?

Example: Dark Matter Phase-in

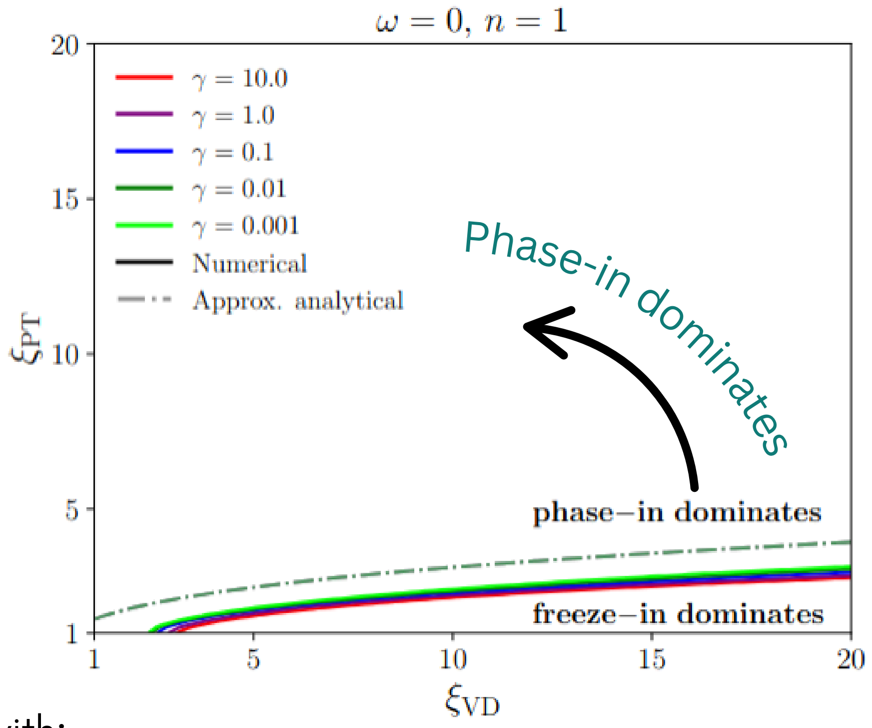
based on: [2504.10593] with C. Benso and F. Kahlhoefer



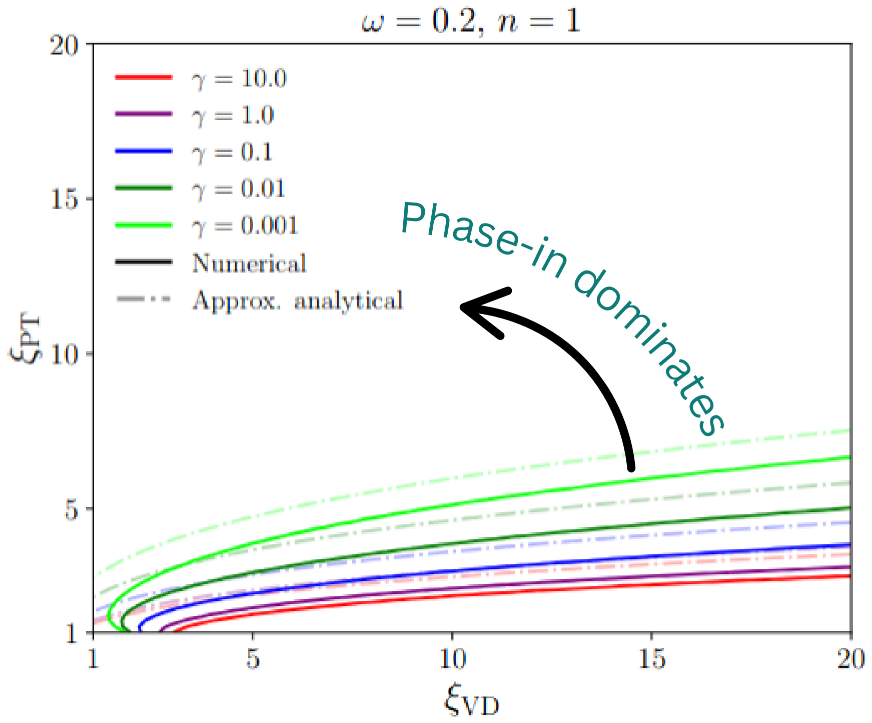
Boltzmann equations for energy/number densities:

$$\frac{\mathrm{d}\rho_\phi}{\mathrm{d}a} = -\frac{3(1+\omega)}{a}\rho_\phi - \frac{\Gamma}{aH}\rho_\phi \qquad \frac{\mathrm{d}\rho_{\rm SM}}{\mathrm{d}a} = -\frac{4}{a}\rho_{\rm SM} + \frac{\Gamma}{aH}\rho_\phi \qquad \frac{\mathrm{d}n_{\rm DM}}{\mathrm{d}a} = -\frac{3}{a}n_{\rm DM} + \frac{\langle\sigma v\rangle}{aH}n_{\rm SM}^2$$

matter dom.

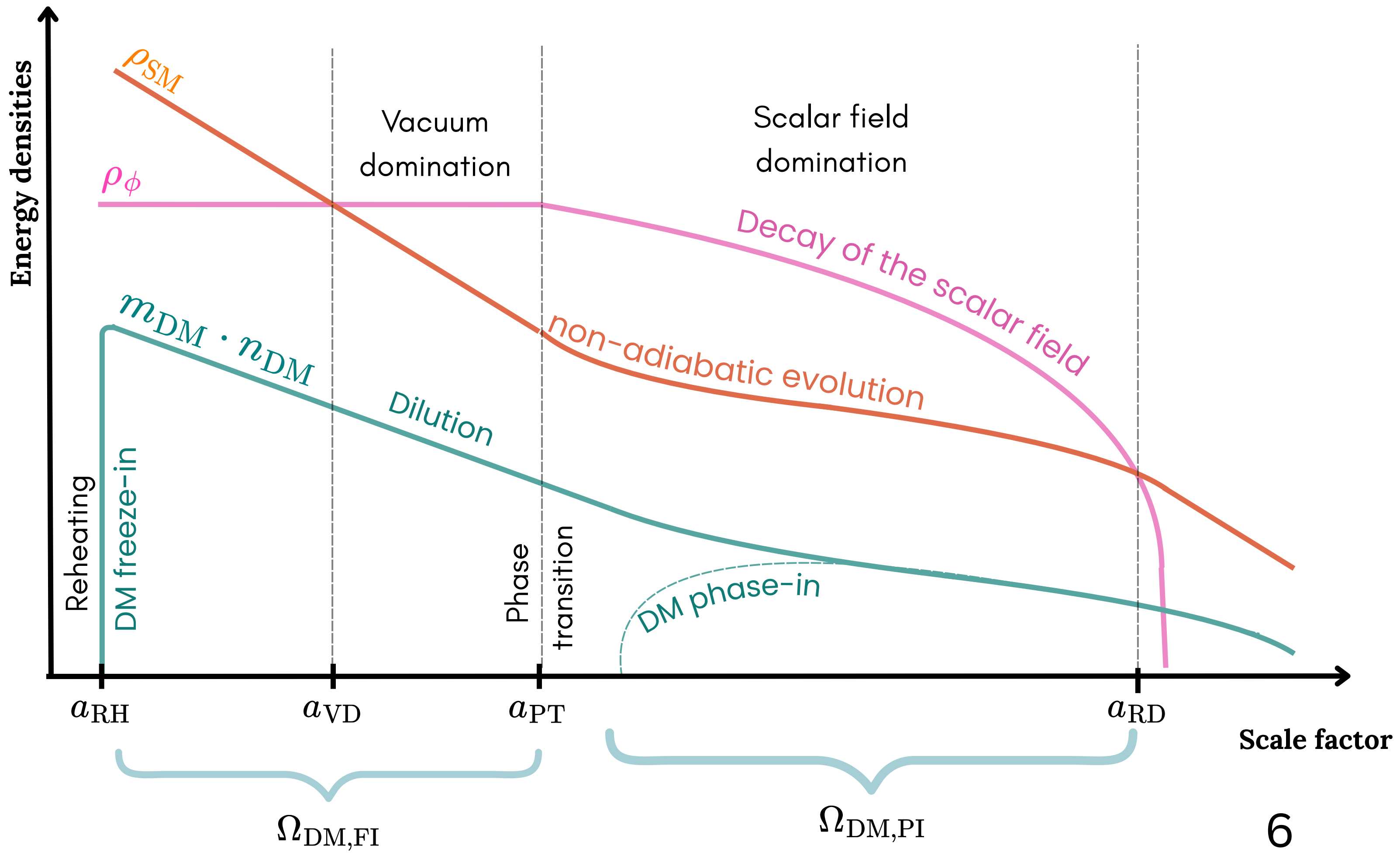


modified cosmology



with:

$\xi_{PT} = \frac{T_{VD}}{T_{PT}}$ (amount of supercooling) $\xi_{VD} = \frac{T_{RH}}{T_{VD}}$ (high/low reheating temp.) $\gamma = \frac{\Gamma}{H(a_{\rm PT})}$ (speed of the decay)



Question: What is the equation of state of the Universe after a supercooled phase transition?

Step I: Analytical understanding ...

- Build some analytical understanding of the equation of state:

General EoS for a inhomogeneous scalar field:

$$\omega = \frac{p_\phi}{\rho_\phi} = \frac{\langle K \rangle - (d-2)/d \, \langle G \rangle - \langle V \rangle}{\langle K \rangle + \langle G \rangle + \langle V \rangle}$$

with $d =$ # of spatial dimensions

kinetic — gradient — potential

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kinetic — gradient — potential

- Assume a generic polynomial potential and focus on the scalar field dynamics (no T dependence):

Equation of motion in expanding background

$$\ddot{\phi} - a^{-2}\vec{\nabla}^2\phi + dH\dot{\phi} = -V_{,\phi}$$

with $V(\phi) = \frac{m_\phi^2}{2}\phi^2 - \kappa\phi^3 + \alpha\phi^4$

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- Apply the virial theorem to determine a relation between the averaged energy densities: After percolation, the solution can be approximated by oscillations of the field

Approximate around the minimum

$$\langle K\rangle \simeq \langle G\rangle + \langle \phi V_{,\phi}/2\rangle + d\left\langle H\phi\dot{\phi}\right\rangle/2 \quad \Rightarrow \quad V(\phi) \sim \phi^2 \quad \Rightarrow \quad \langle K\rangle \simeq \langle G\rangle + \langle V\rangle$$

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KIT Karlsruher Institut für Technologie

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$$\langle K\rangle \simeq \langle G\rangle + \langle \phi V_{,\phi}/2\rangle + d\left\langle H\phi\dot{\phi}\right\rangle/2 \quad \Longrightarrow \quad V(\phi)\sim\phi^2 \quad \Longrightarrow \quad \langle K\rangle \simeq \langle G\rangle + \langle V\rangle$$

gradient ($\langle G\rangle$), potential ($\langle V\rangle$)

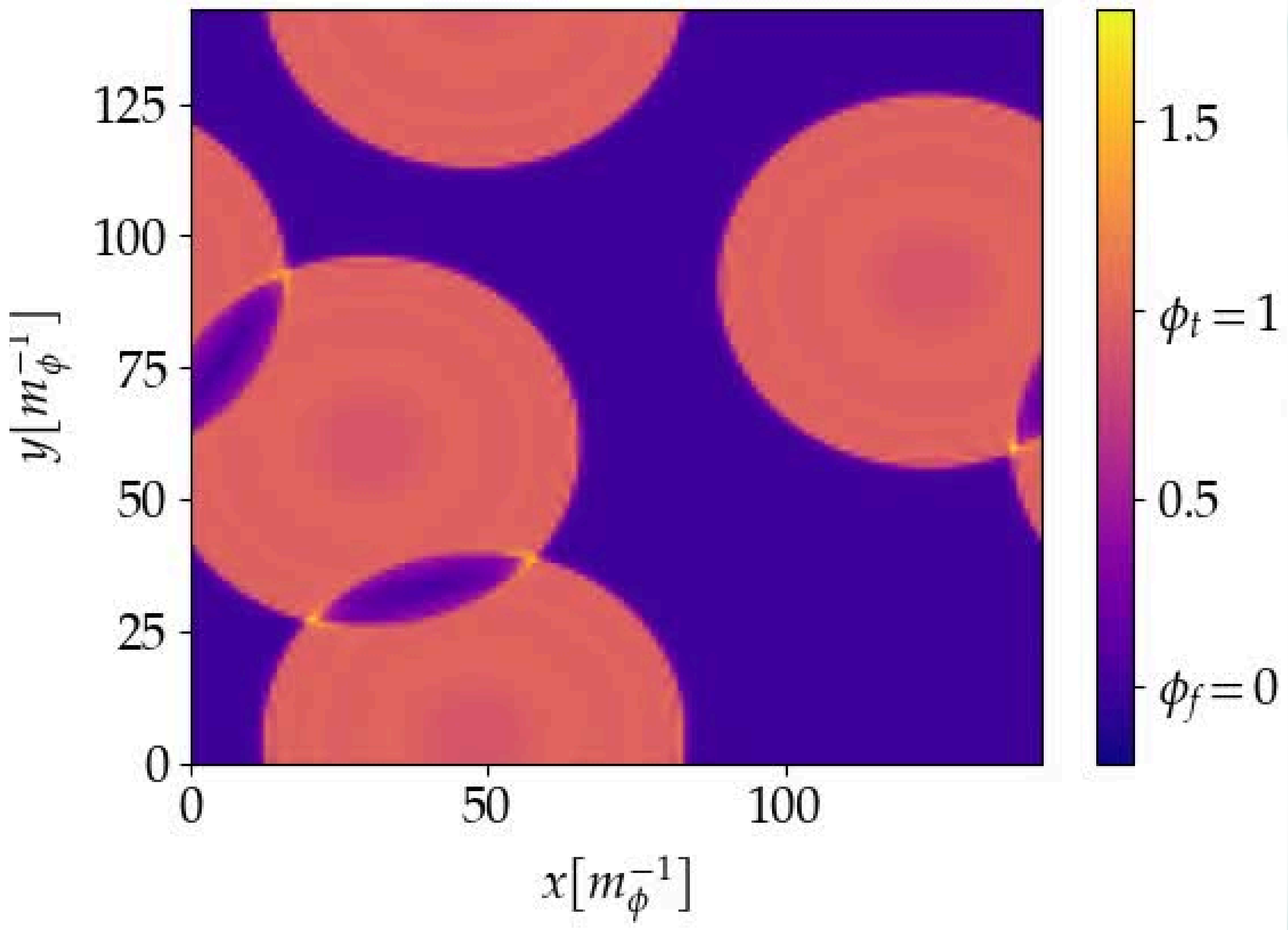
- We find a simplified expression:

$$\omega = d^{-1}\frac{\langle G\rangle}{\langle G\rangle + \langle V\rangle}$$

Step II : Let a computer do it

Solve on the lattice: $\ddot{\phi} - a^{-2}\vec{\nabla}^2\phi + dH\dot{\phi} = -V_{,\phi}$ with $V(\phi) = \frac{m_\phi^2}{2}\phi^2 - \kappa\phi^3 + \alpha\phi^4$

Example: 2D simulation with random nucleation and static background



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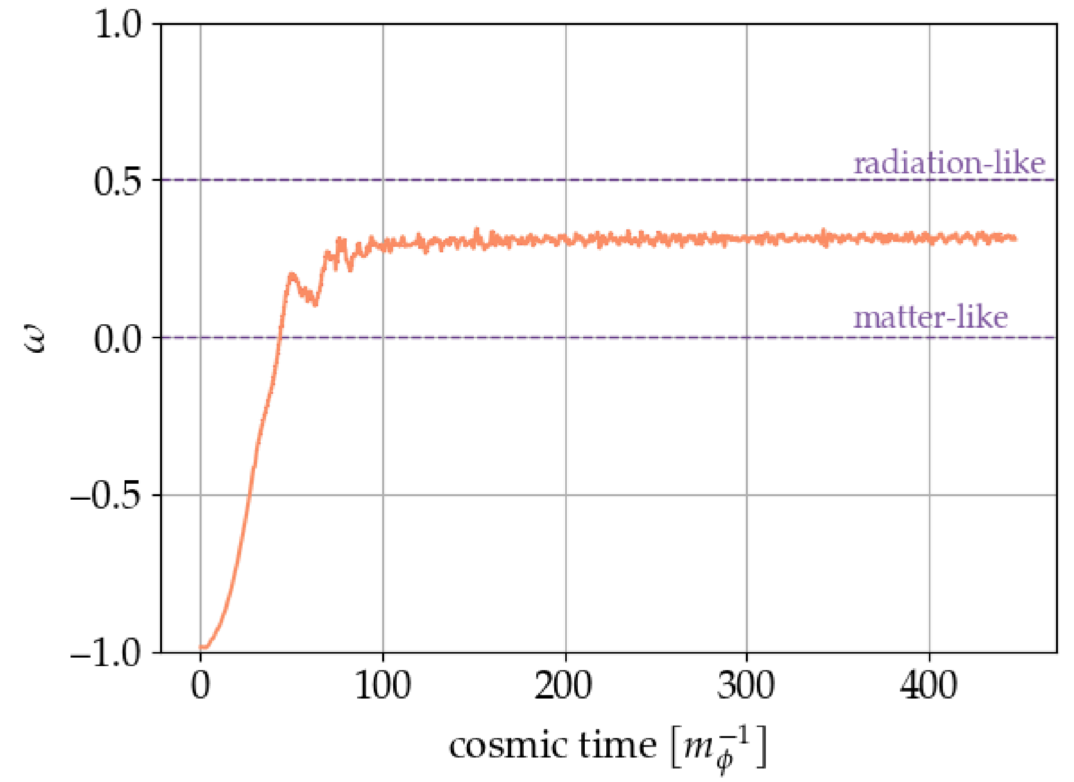
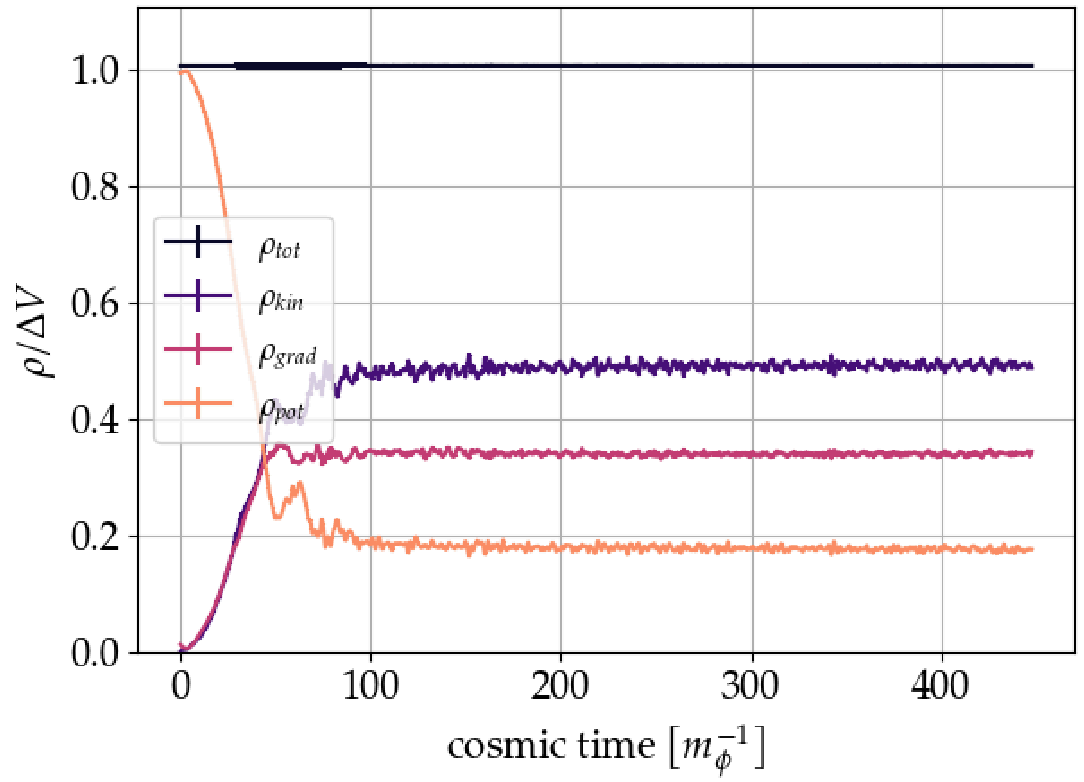
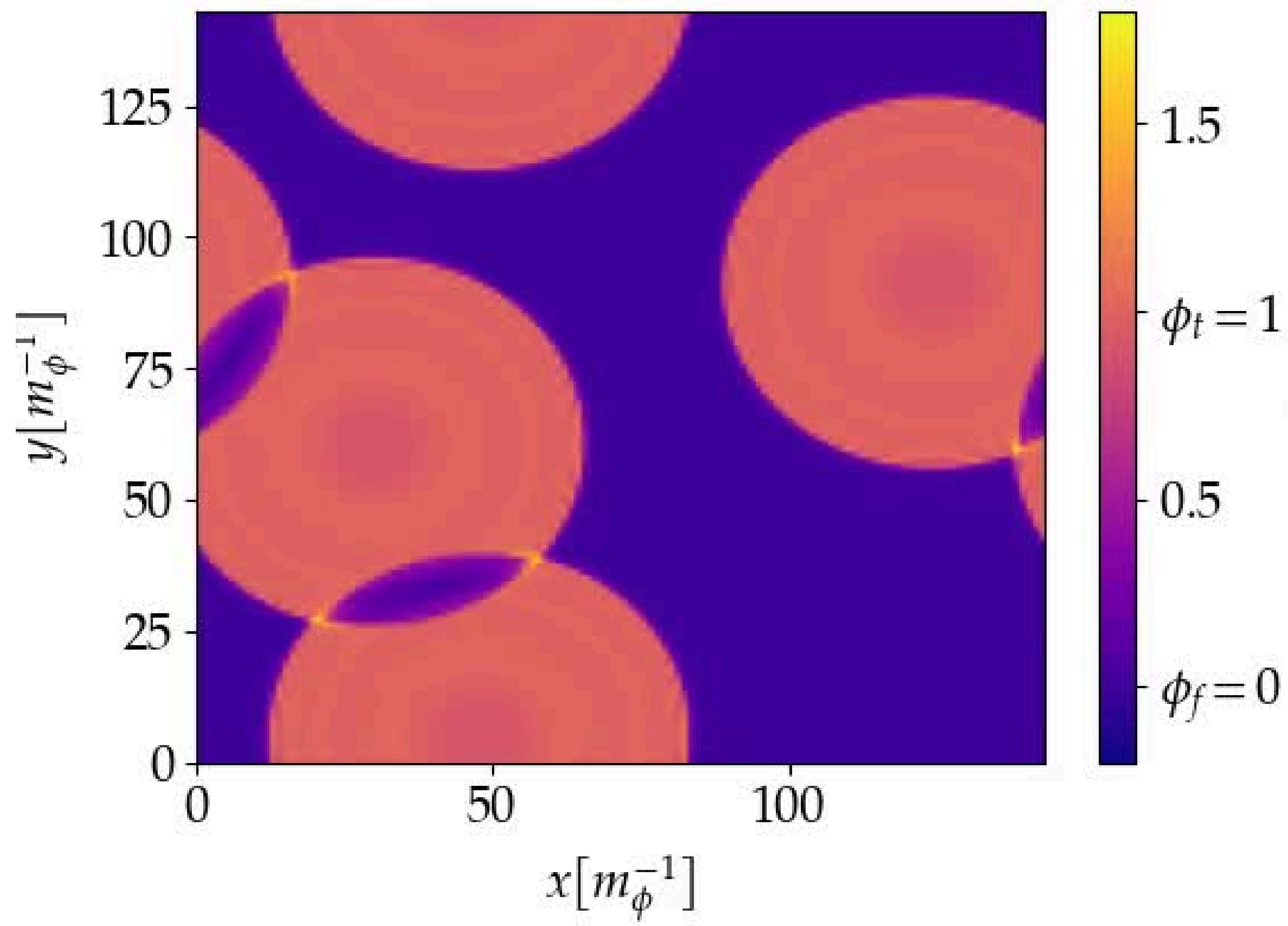
The EoS can be determined from the evolution of the energy densities: $\omega = \frac{p_\phi}{\rho_\phi} = \frac{\langle K\rangle - (d-2)/d\,\langle G\rangle - \langle V\rangle}{\langle K\rangle + \langle G\rangle + \langle V\rangle}$





Example: 2D simulation with random nucleation and static background



Virial Theorem

$\langle K\rangle \simeq \langle G\rangle + \langle V\rangle$ is fullfilled!

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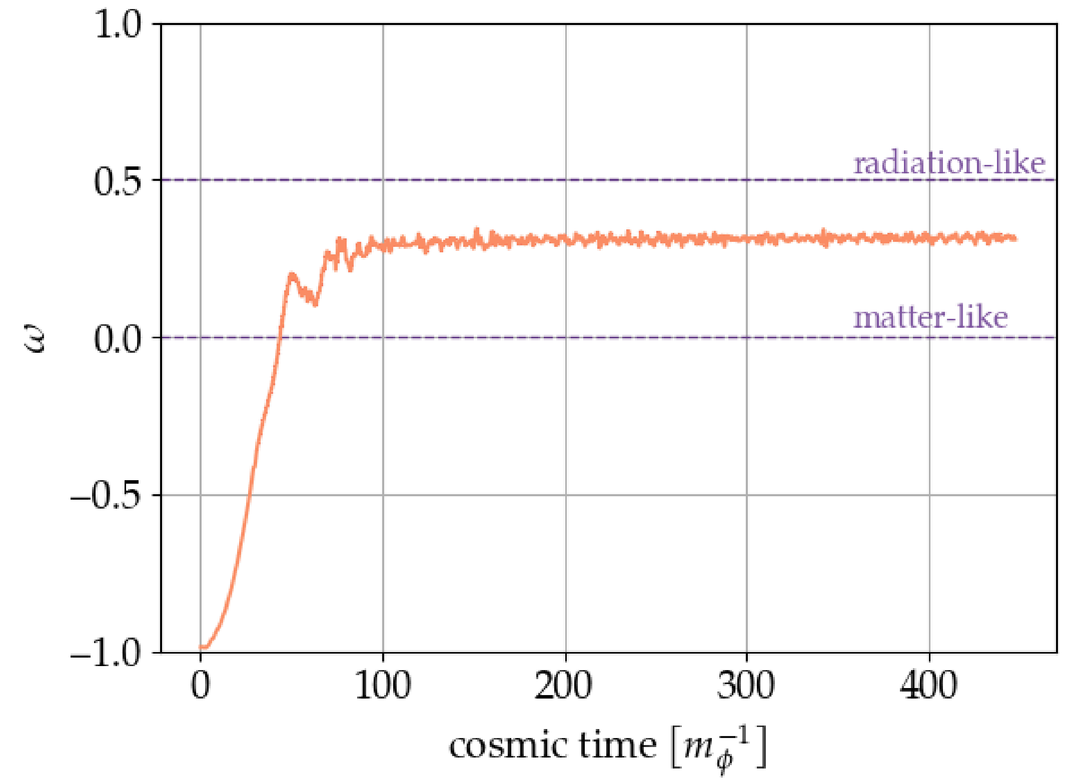

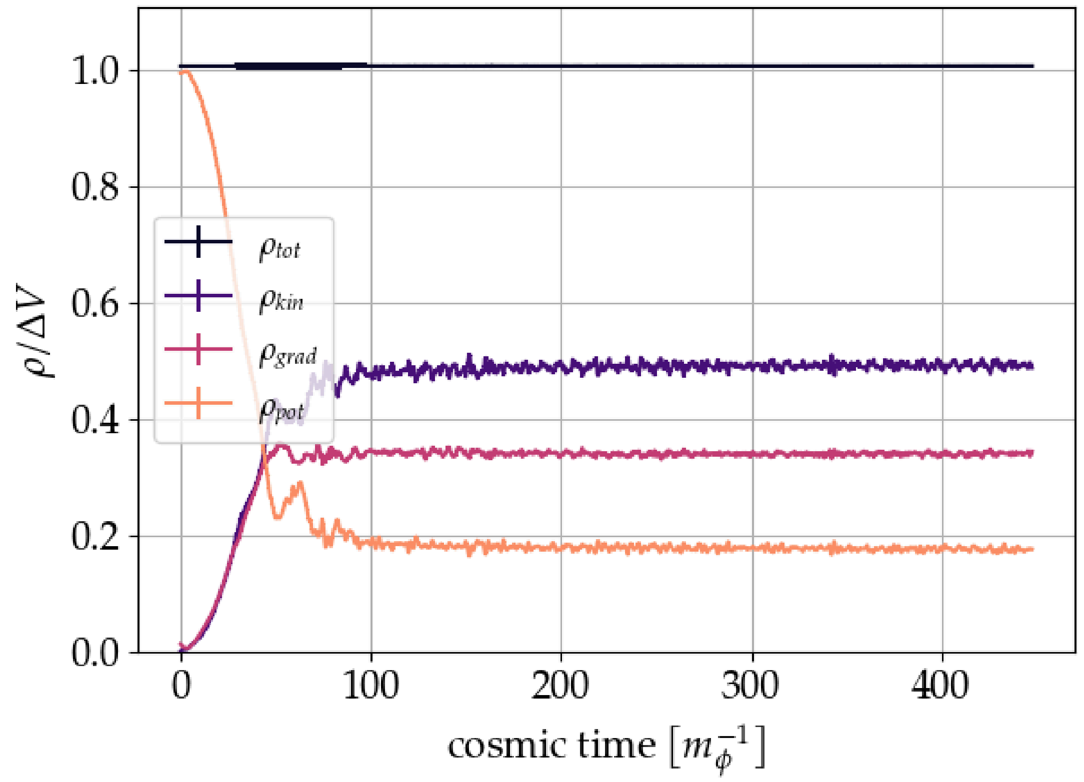

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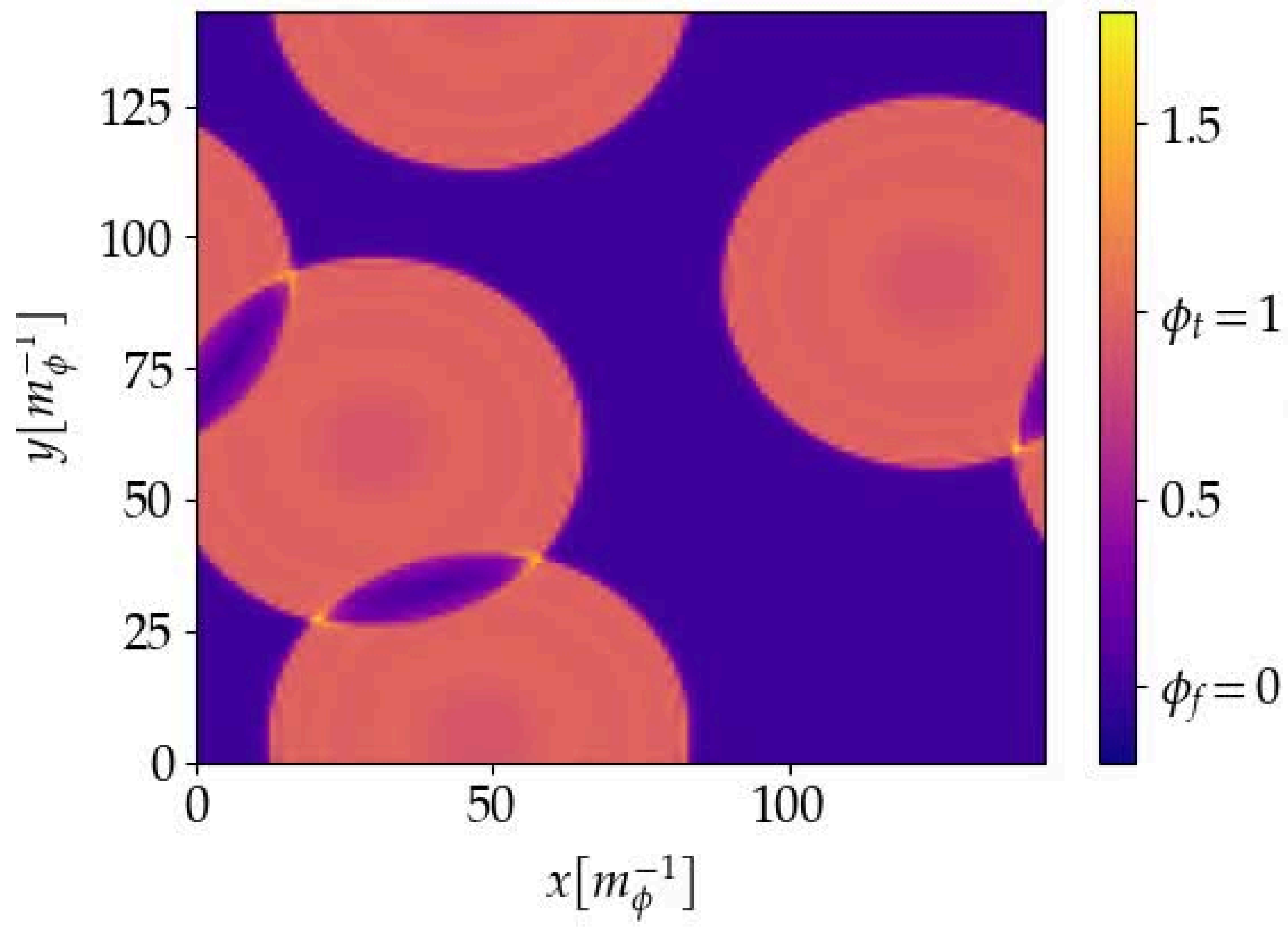


The lattice results show an equation of state value between matter and radiation domination. What does the EoS depend on?

What determines the EoS?

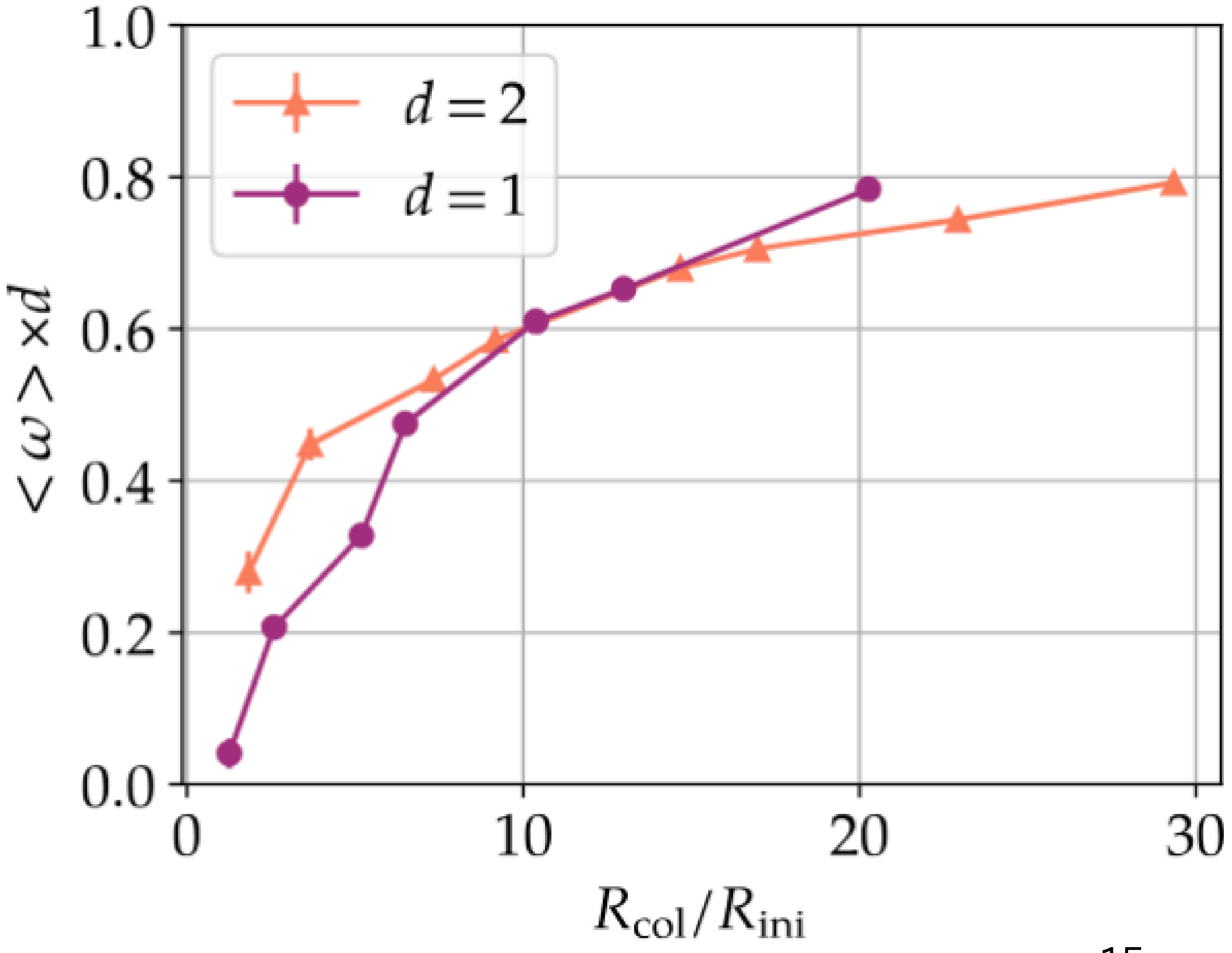
(Step III: Let a computer do it again and again)

Systematic study of the simulation results for bubble collisions in 1, 2 and 3 spatial dimensions

Varying the initial bubble separation to reach different bubble wall velocities:

Larger bubble separation
→ more relativistic bubble walls
→ more energy in gradients
→ closer to radiation domination

Summarized results for the EoS after percolation (1+1 and 1+2 simulations)



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What about predictivity? Is it possible to determine a function $\omega(\gamma_\star)$?

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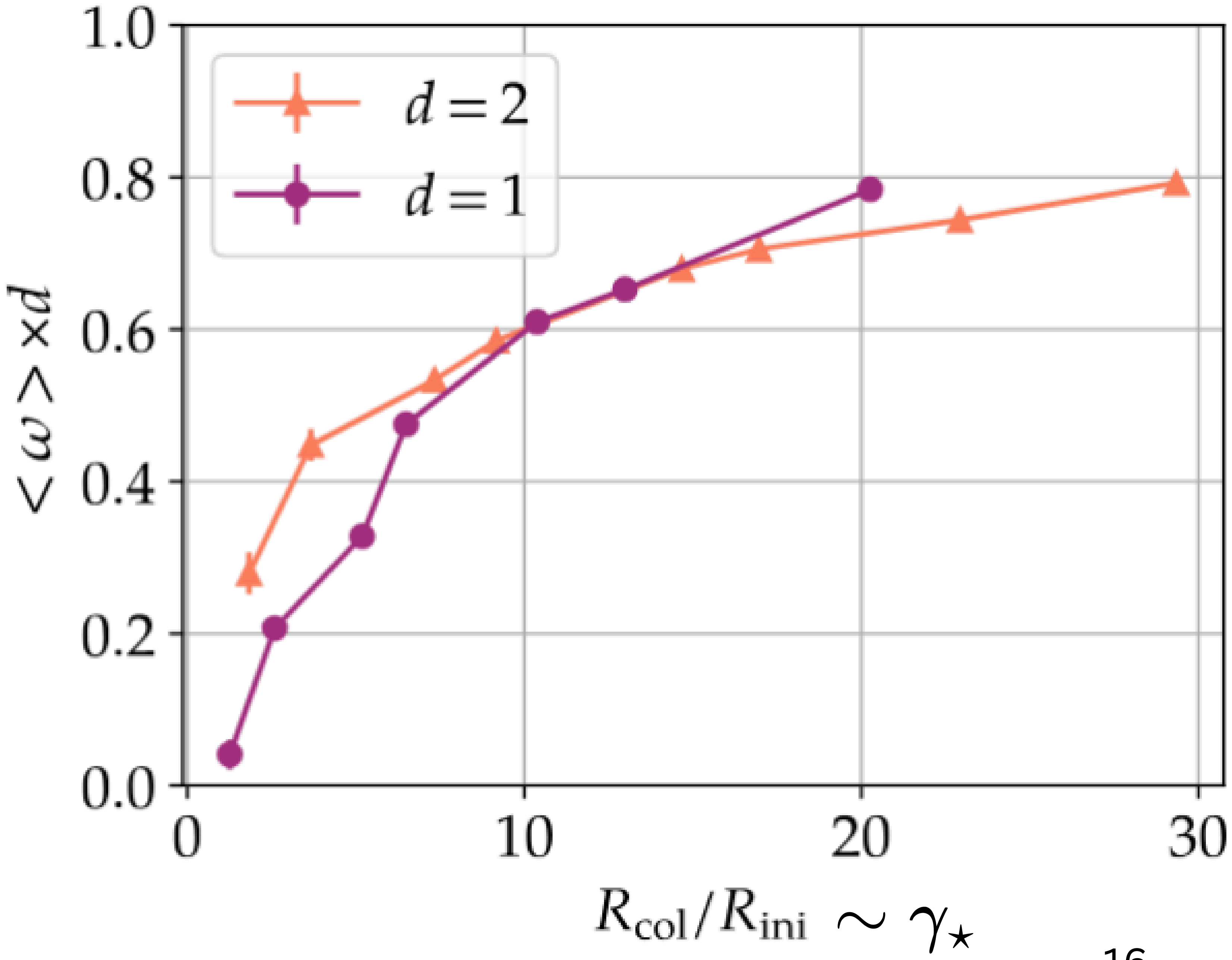
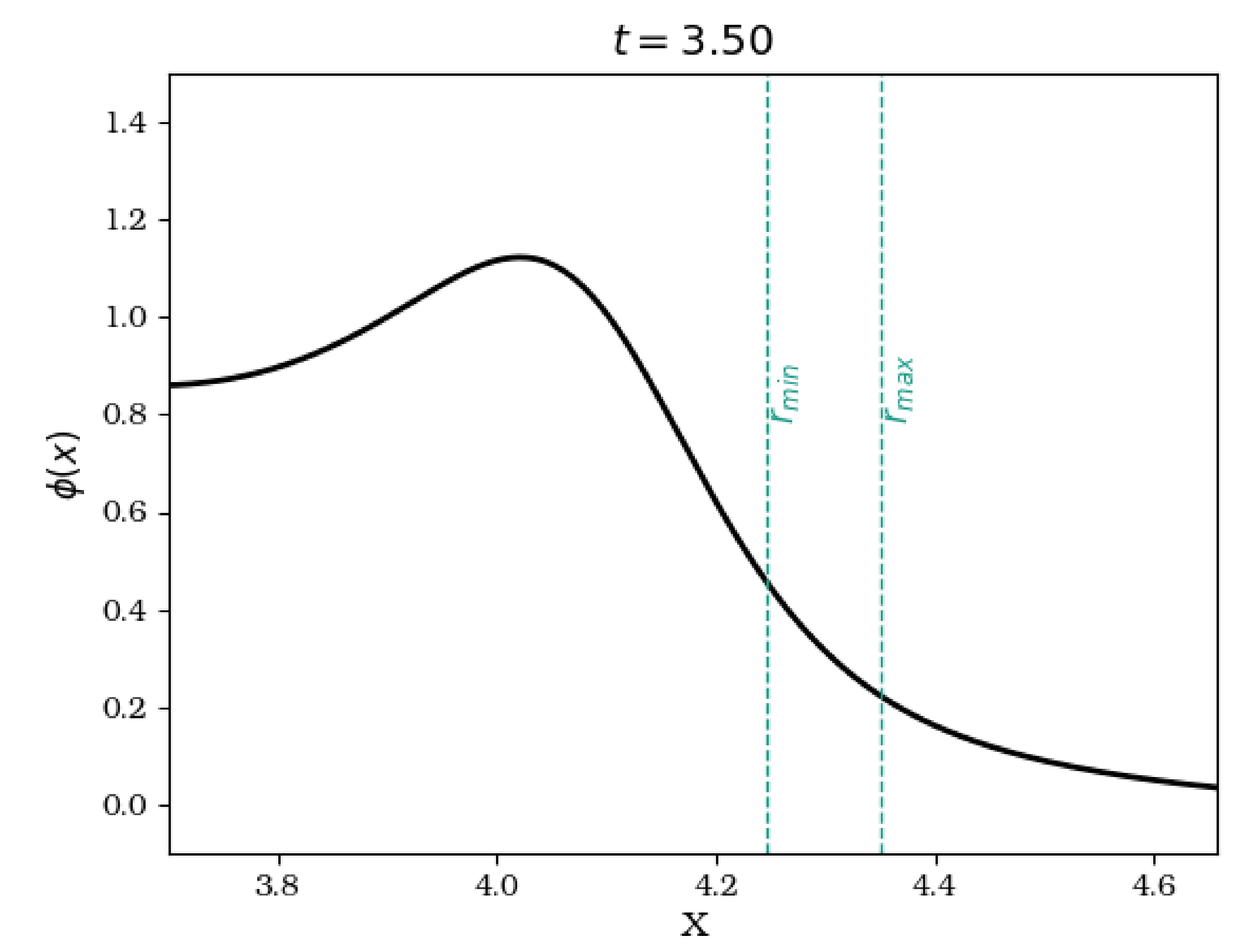
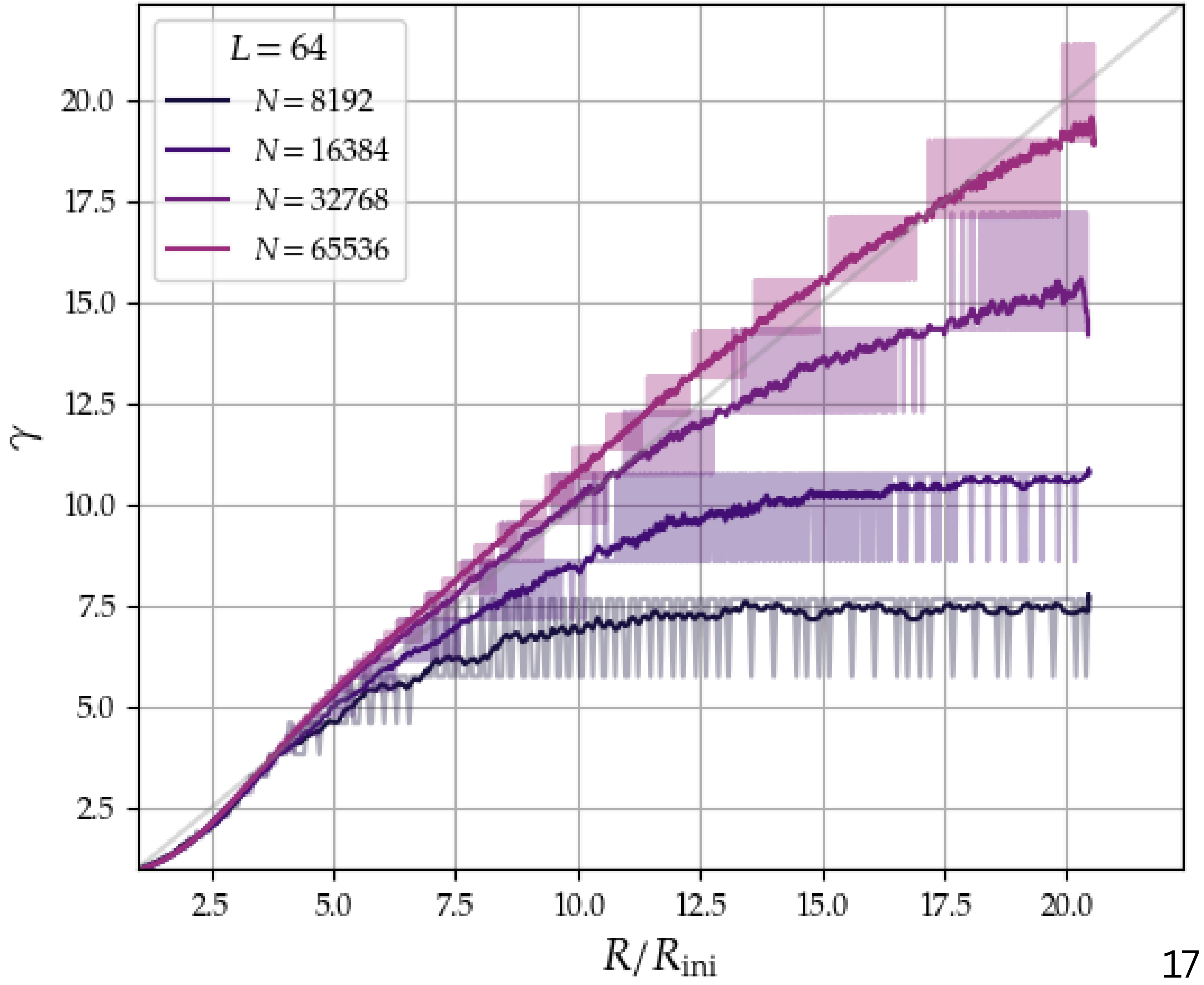


Determination of Lorentz factor

We track the wall thickness during the expansion of 1D bubbles to confirm the relation between γ and R_{coll}/R_{ini} .





Step IV: develop more analytical understanding...

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Move to fourier space and re-write the previous expressions in terms of the power spectrum

$$\langle \phi_{\mathbf{k}'} \phi_{\mathbf{k}} \rangle = P_\phi(k)(2\pi)^3 \delta^{(3)}(\mathbf{k} + \mathbf{k}')$$

We introduce the dimensionless power spectrum

$$\Delta_\phi(k) = \frac{k^d}{(2\pi)^d} S_{d-1} P_\phi(k), \quad \text{with} \quad S_{d-1} = 2\pi^{d/2}/\Gamma\left(d/2\right)$$

With some simplifications, one can re-express the energy densities in terms of the power spectrum:

$$\omega = d^{-1} \frac{\langle G \rangle}{\langle K \rangle} = d^{-1} \frac{\int d\ln k (k/a)^2 \Delta_\phi(k)}{\int d\ln k \left(m_\phi^2 + (k/a)^2 \right) \Delta_\phi(k)}$$

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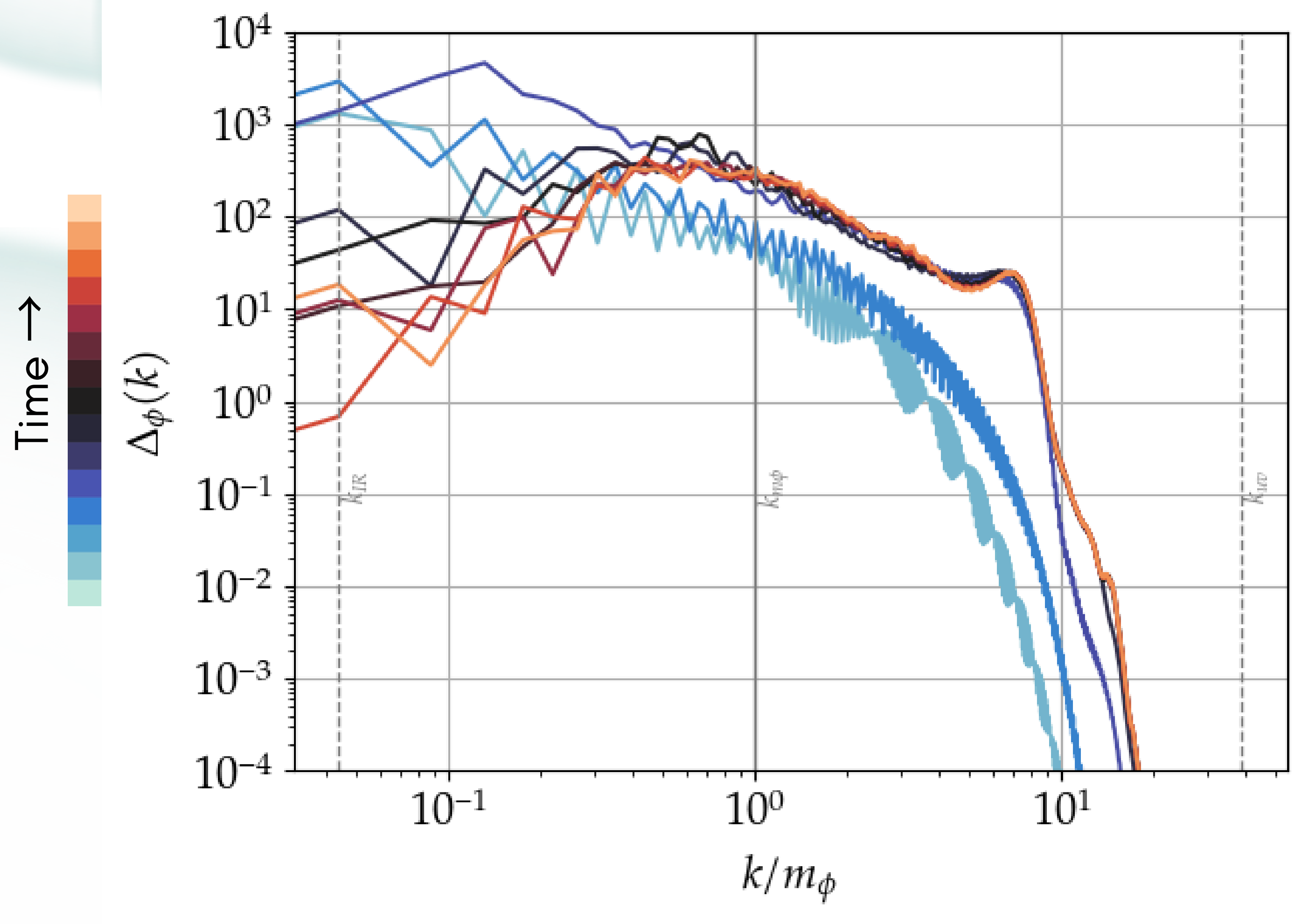
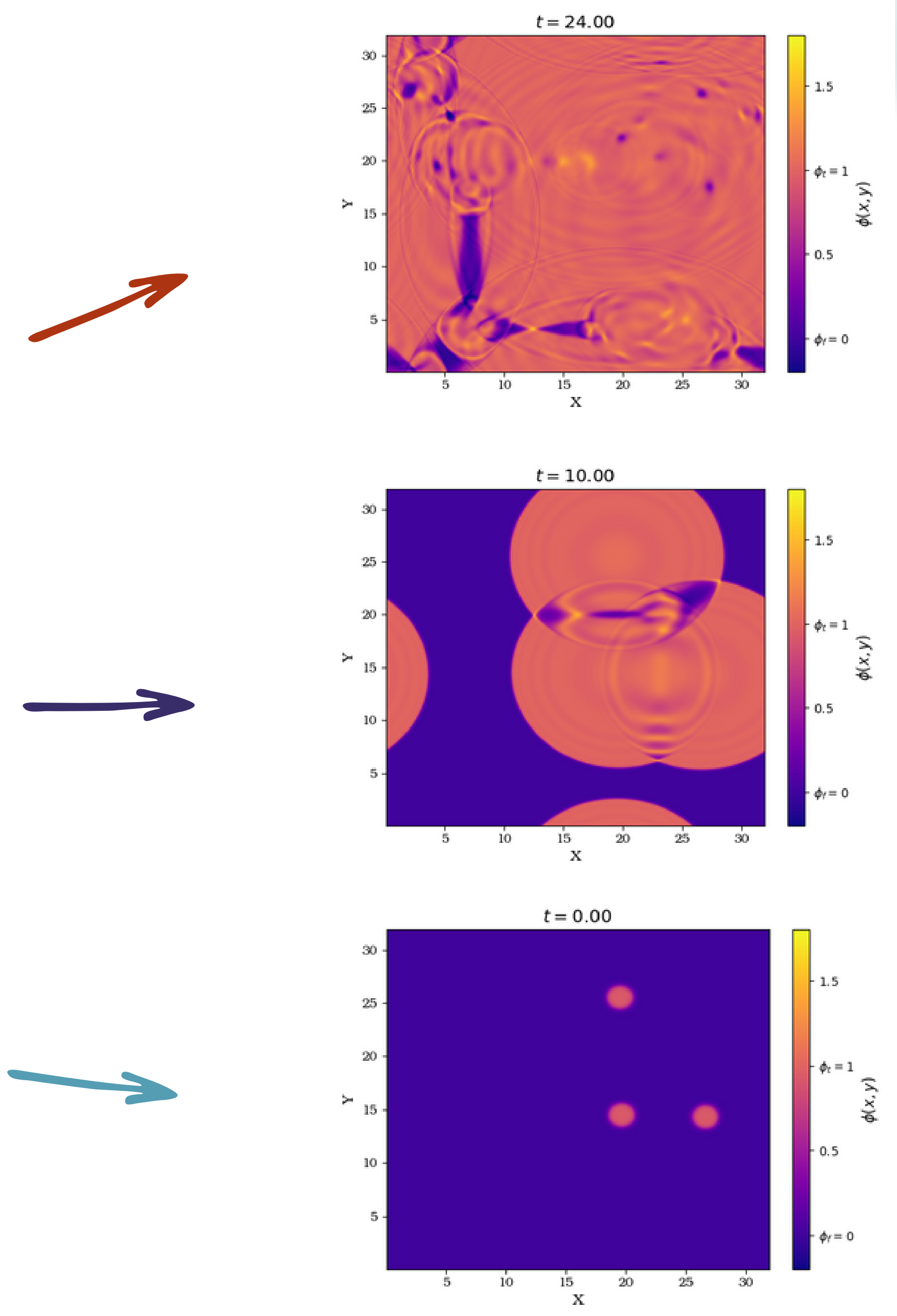
Is it possible to determine a function $\omega(\gamma_\star)$?

Maybe if we find determine an analytic form for the dimensionless PS in terms of the lorentz factor of the wall γ

Step V: Study the power spectrum

Evolution of the power spectrum before and after collision/percolation (results of 3D simulation)





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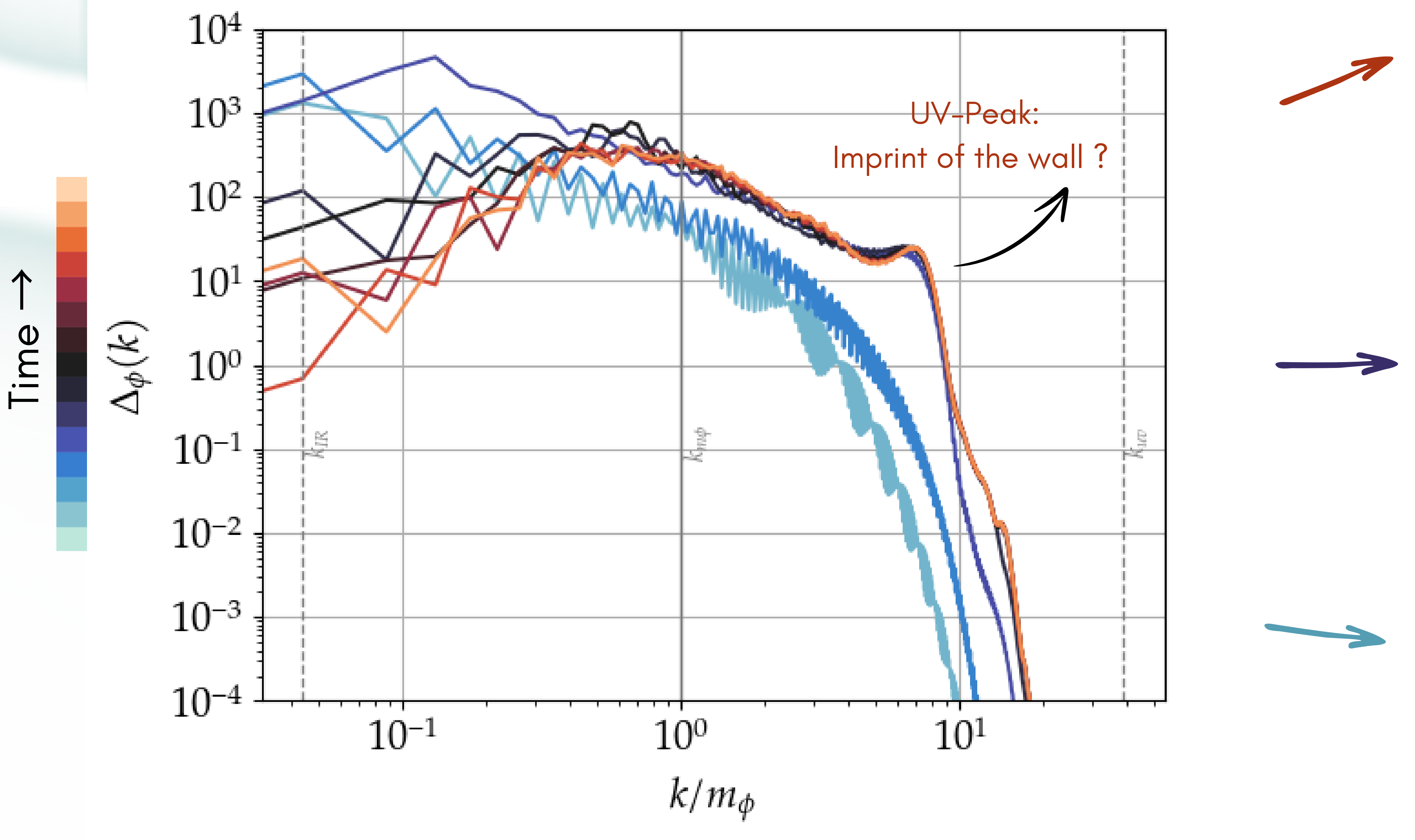

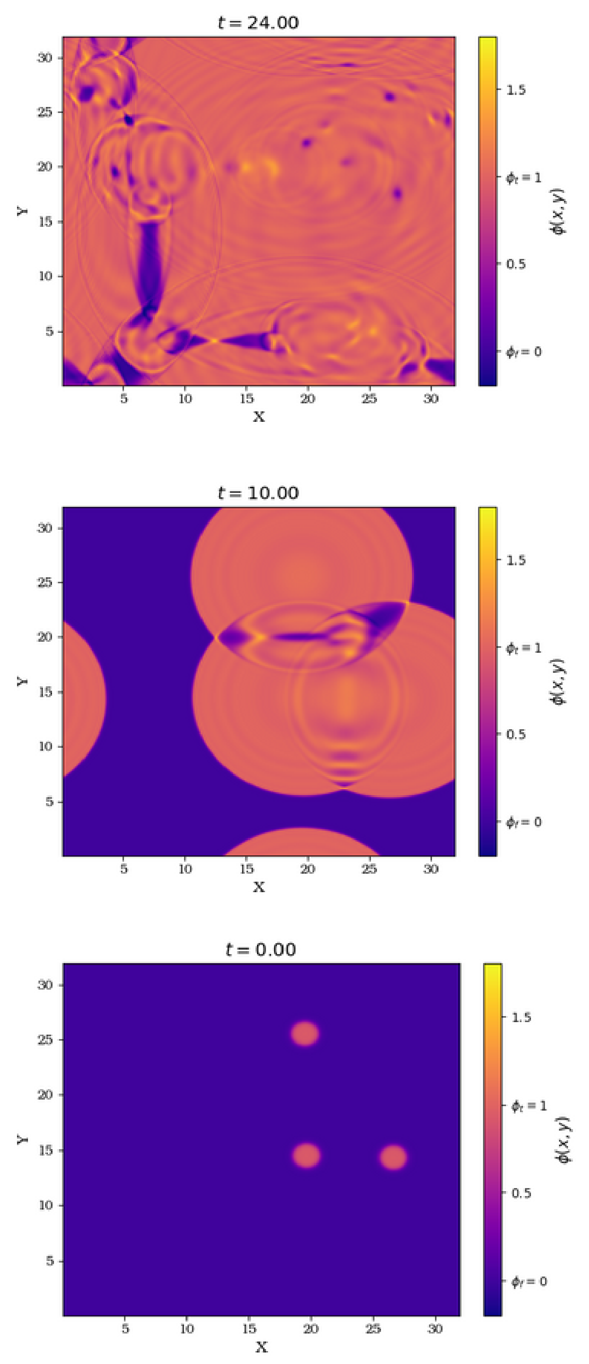

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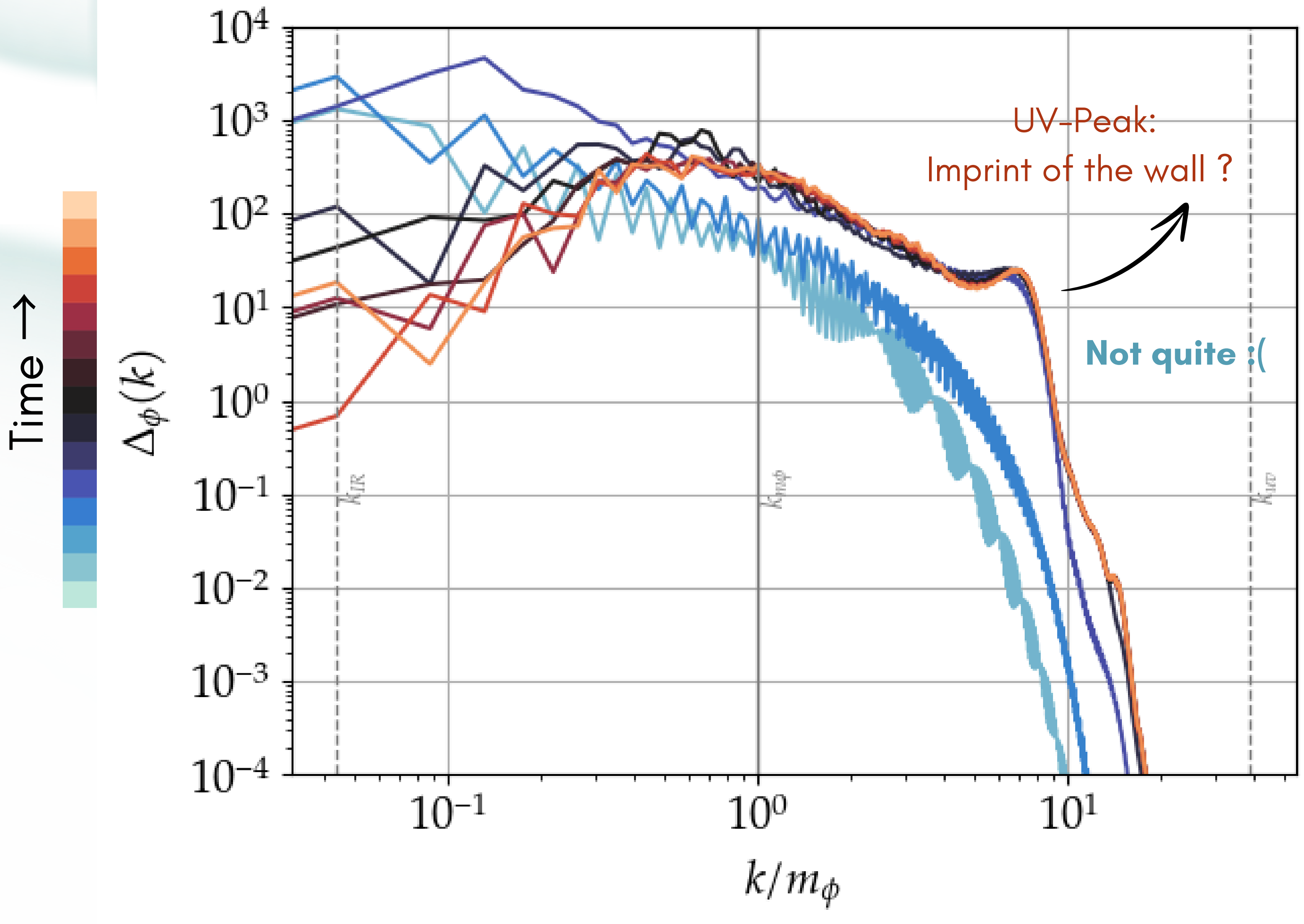
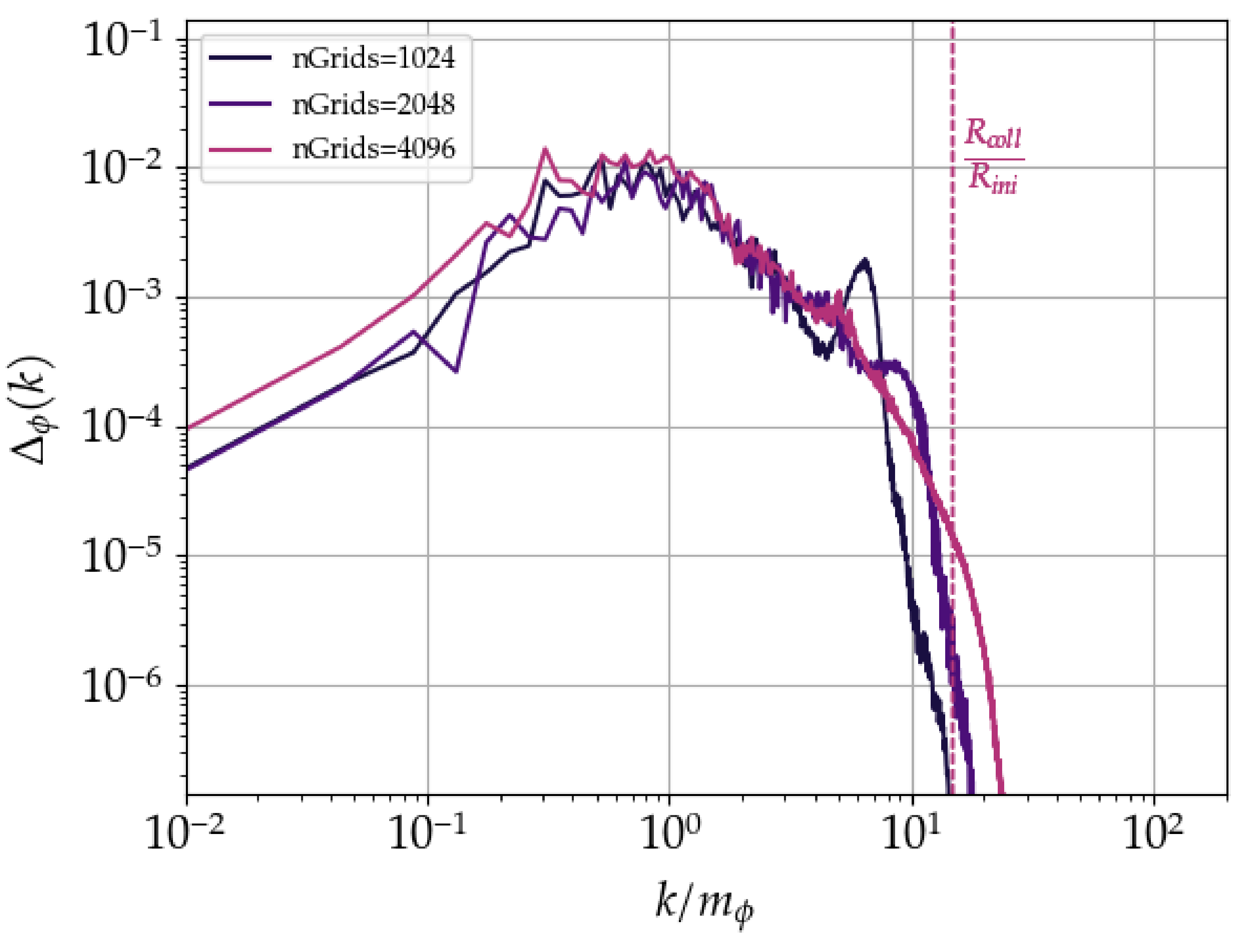


We have performed checks using 2D simulations. These show that the peak dissapears with higher resolution.



The peak seems to be a lattice artifact. The power spectrum follows instead a simple broken power law.

Conclusions and outlook

Immediately after a first order phase transition we find that:
Larger bubble separation
→ more relativistic bubble walls
→ more energy in gradients
→ closer to radiation domination

An analytically motivated function for $\omega(\gamma_\star)$ will follow.

If thermalization is very slow, the expansion will eventually suppress the gradients and might still allow for a purely matter-dominated epoch.
Detailed results on the evolution of the EoS with expansion are upcoming.