The Geometry of Interactions - Curved Field Space of Scalars at 1-Loop

- Dominik Haslehner

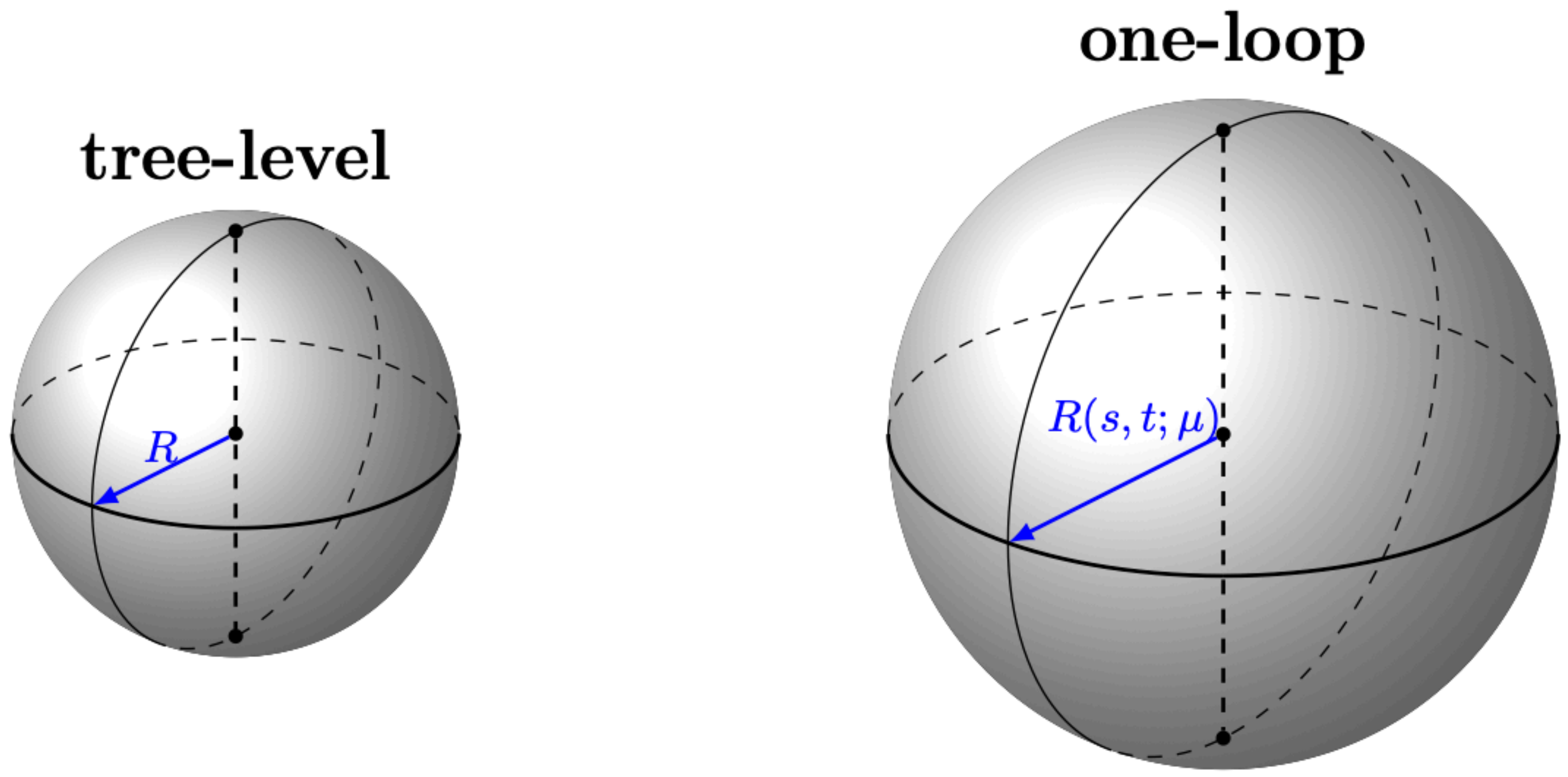
- Based on JHEP 07 (2025) 167; (arXiv: 2503.09785)

  In collaboration with Patrick Aigner, Luigi Bellafronte, Emanuele Gendy and Andreas Weiler

Scalars, Warsaw 2025

Main Point:

Curvature of the Field-Space Manifold changes as a function of the renormalisation scale and the scattering kinematics



$$R \longrightarrow R(s, t; \mu)$$

Outline

- Introduction: Field Space Geometry
- Going to 1-Loop in the NLSM
- Geometry-Kinematics Duality
- Curvature Renormalisation
- Conclusion

Field-Space Geometry

- Given $\mathscr{L}$ of Non-Linear Sigma Model

$$\mathscr{L} = \frac{1}{2} g_{ab}(\phi)\, \partial^\mu \phi^a\, \partial_\mu \phi^b$$

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with $g_{ab}(\phi) = \delta + \#\phi + \#\phi^2 + \dots$

Field-Space Geometry

- Given $\mathscr{L}$ of Non-Linear Sigma Model
- Invariant under Field redefinitions

$$\mathscr{L} = \frac{1}{2} g_{ab}(\phi)\, \partial^\mu \phi^a \, \partial_\mu \phi^b$$

with $g_{ab}(\phi) = \delta + \#\phi + \#\phi^2 + \ldots$

Set of all

Field-Redefinitions

Field-Space Geometry

- Given $\mathscr{L}$ of Non-Linear Sigma Model

$$\mathscr{L} = \frac{1}{2}\, g_{ab}(\phi)\, \partial^\mu \phi^a\, \partial_\mu \phi^b$$

- Consider a subset of all Field redefinitions

Subset under which:

$$\partial_\mu \phi^a \;\to\; \partial_\mu \tilde{\phi}^a = \frac{\partial \tilde{\phi}^a}{\partial \phi^b}\, \partial_\mu \phi^b$$

$$g_{ab}(\phi) \;\to\; \tilde{g}_{ab}(\tilde{\phi}) = \frac{\partial \phi^c}{\partial \tilde{\phi}^a} \frac{\partial \phi^d}{\partial \tilde{\phi}^b}\, g_{cd}(\phi)$$

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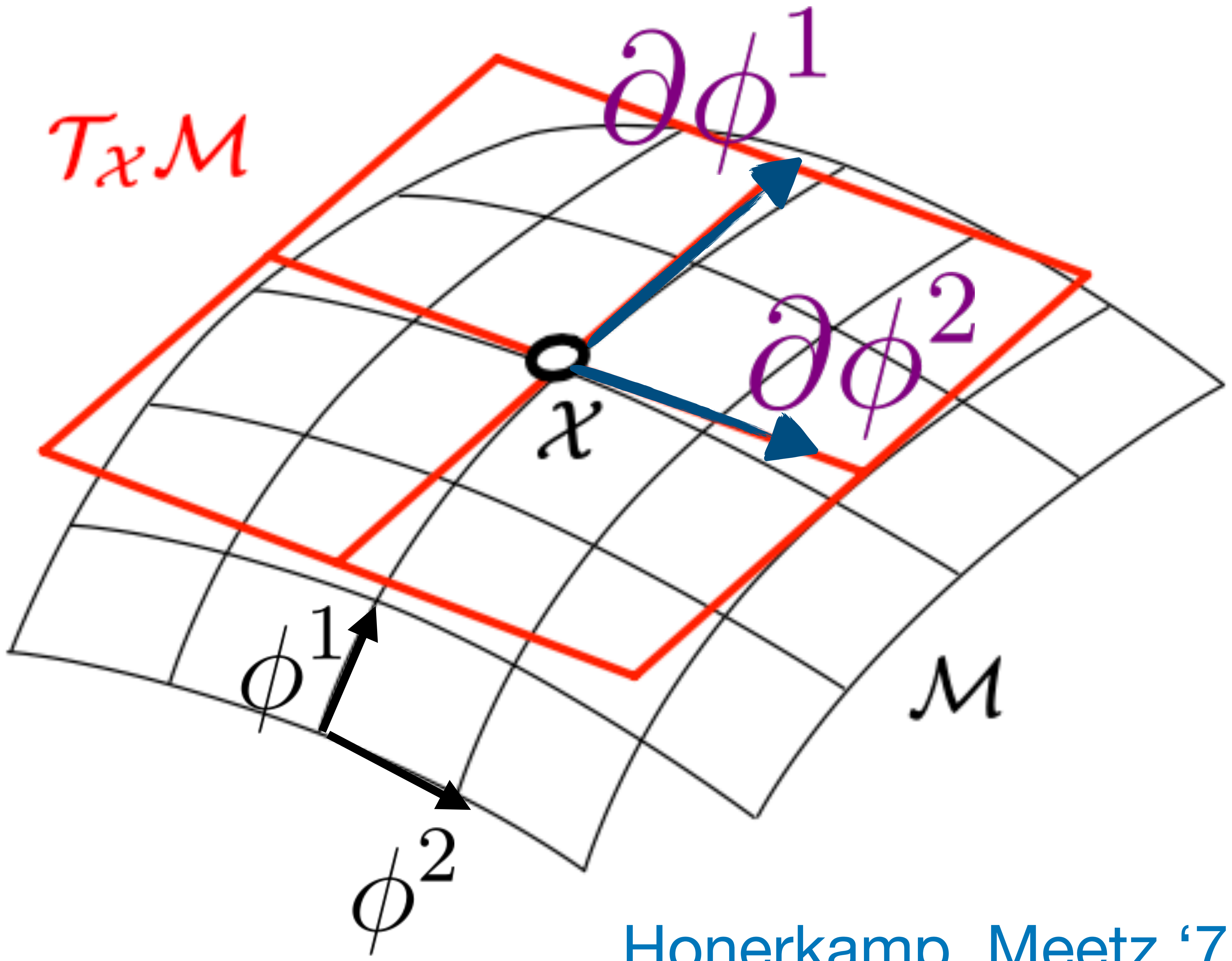
Field-Space Geometry

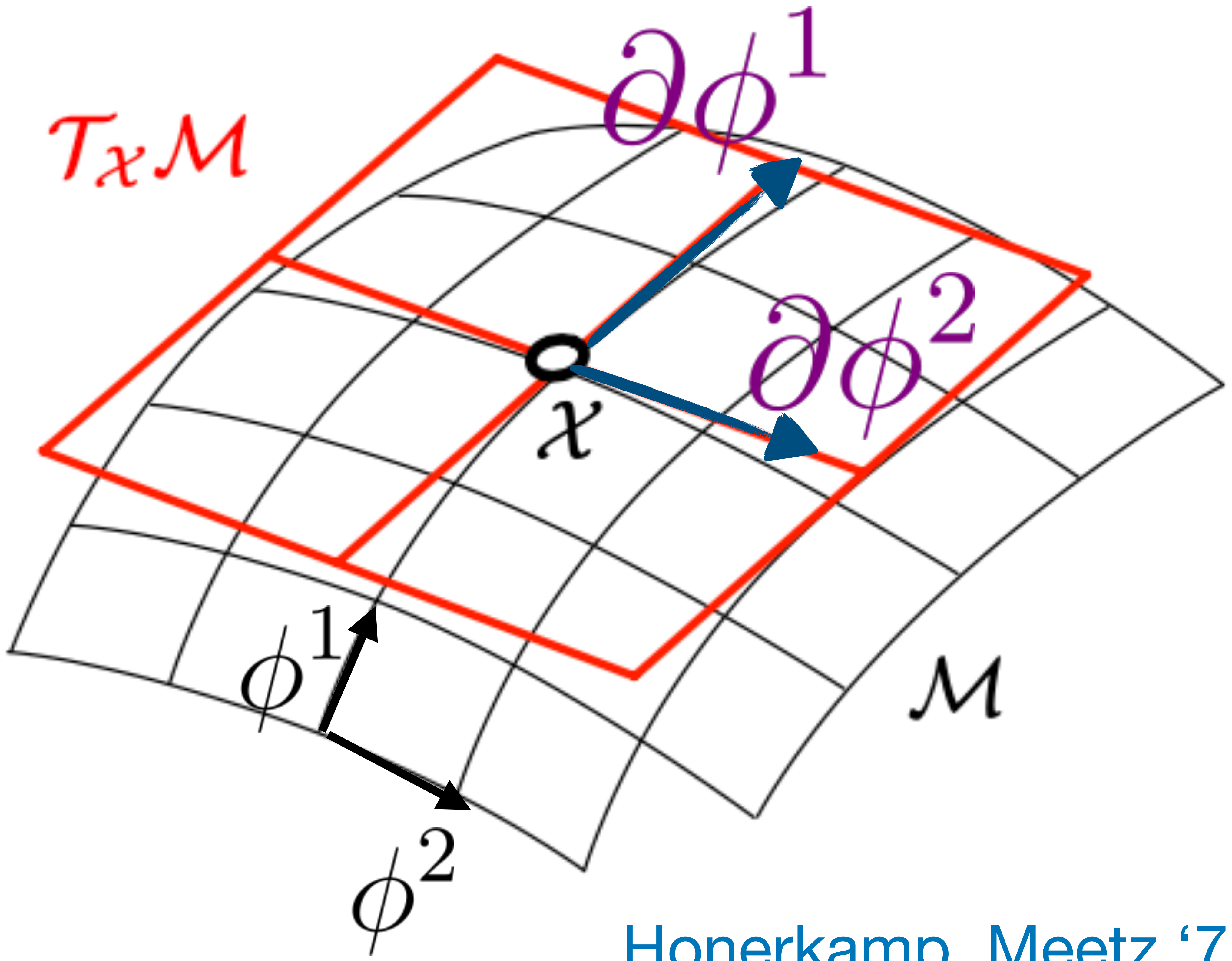
- **Riemannian Manifold $\mathscr{M}$**
  - ϕ^a: coordinates on $\mathscr{M}$
  - $\partial\phi^a$: tangent vectors
  - $g_{ab}(\phi)$: metric on $\mathscr{M}$



Honerkamp, Meetz ‘71
Volkov ’73

Field-Space Geometry

- **Riemannian Manifold $\mathscr{M}$**
  - ϕ^a: coordinates on $\mathscr{M}$
  - $\partial\phi^a$: tangent vectors
  - $g_{ab}(\phi)$: metric on $\mathscr{M}$
- We call $\mathscr{M}$ the **Field-Space Manifold**
  - $\dim \mathscr{M} = \#\,$(scalar fields)



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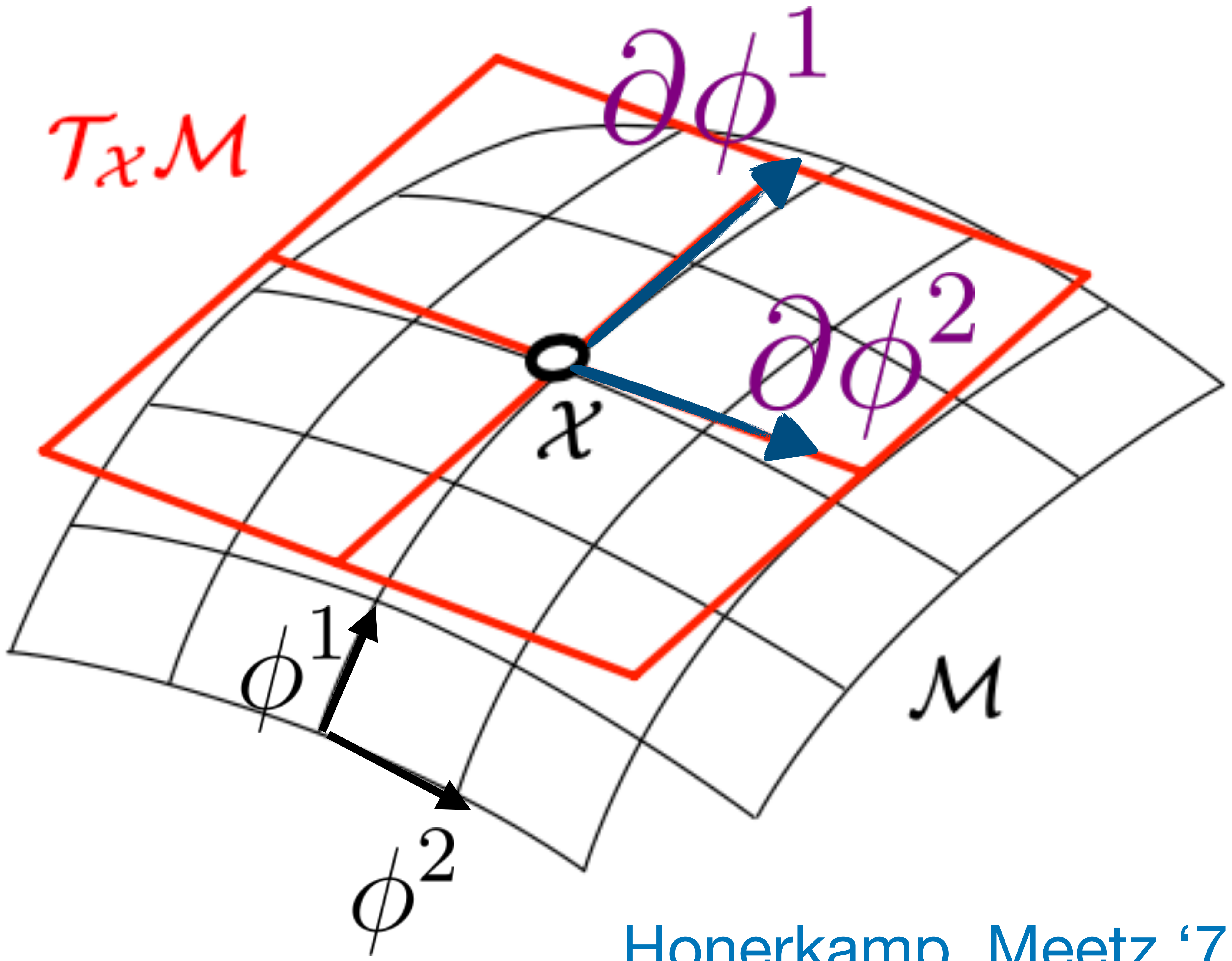
Field-Space Geometry

- **Field-Space has Riemannian Structure**
  - Can compute geometric quantities
    - $\Gamma^a{}_{bc}, R_{abcd}, \nabla_e R_{abcd}, \ldots$



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Field-Space Geometry

- **Field-Space has Riemannian Structure**
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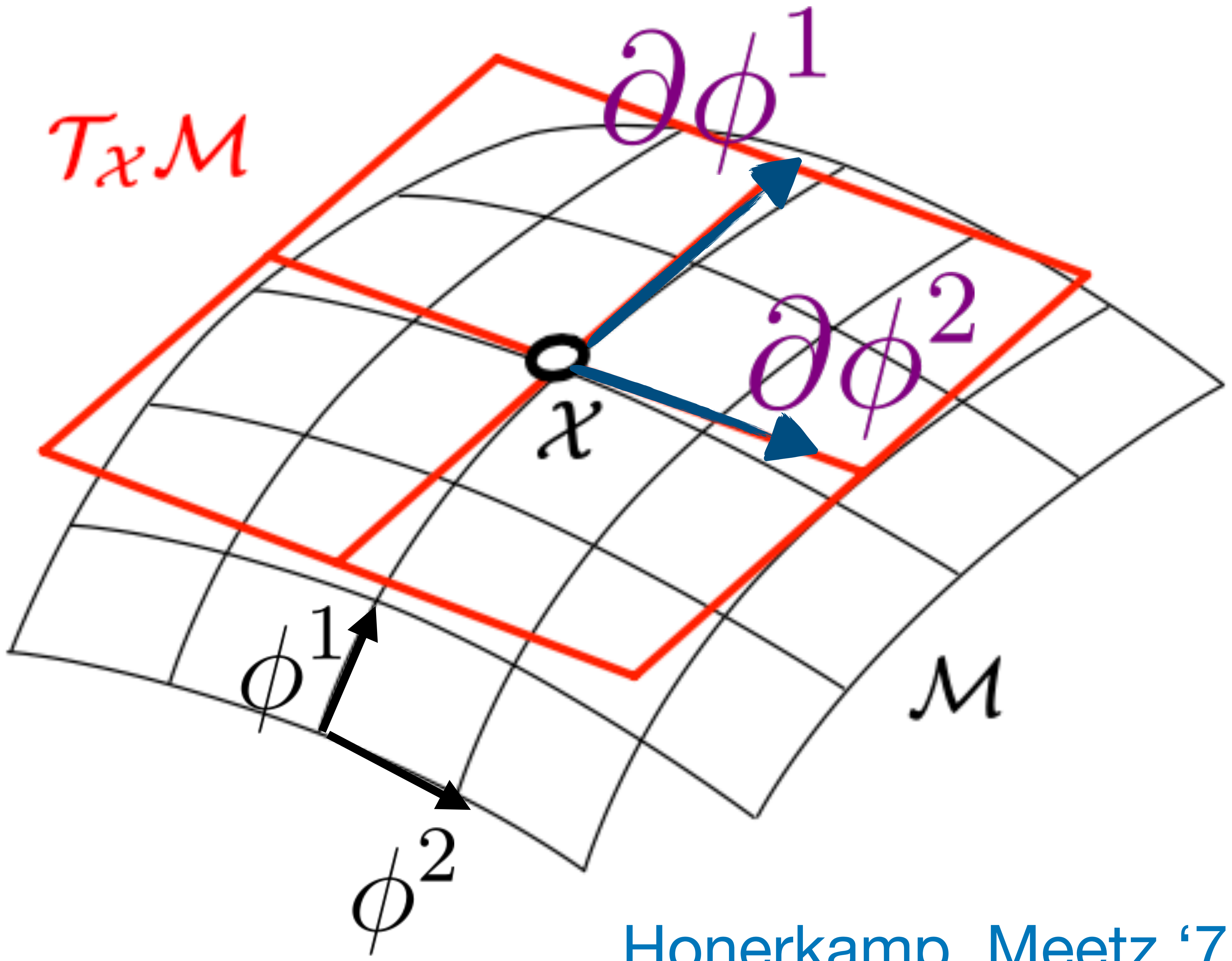
- We can go to Riemann normal coordinates

$$\mathscr{L} = \frac{1}{2}\left(\delta_{ab} + \frac{1}{3}R_{acbd}\phi^c\phi^d + \ldots\right)\partial\phi^a\partial\phi^b$$



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Curvature encodes Interactions

in RNC: $$\mathscr{L} = \frac{1}{2}\left(\delta_{ab} + \frac{1}{3}R_{acbd}\phi^c\phi^d + \dots\right)\partial\phi^a\partial\phi^b$$

Feynman Rules:
($4^{pt.}$ tree level) $$\mathscr{F}_{abcd} \sim f_s(p_i \cdot p_j)\, R_{a(cd)b} + f_a(p_i \cdot p_j)\, R_{a[cd]b}$$

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On-Shell Amplitudes:

$$\mathscr{A}^4{}_{abcd} = R_{abcd}\, t + R_{acbd}\, s$$

$$\mathscr{A}^5{}_{abcde} = \nabla_c R_{adbe}\, s_{45} + \nabla_d R_{acbe}\, s_{35} + \dots$$

- Geometry of the field-space determines interactions and amplitudes

Universal Form

in RNC: $$\mathscr{L} = \frac{1}{2}\left(\delta_{ab} + \frac{1}{3}R_{acbd}\phi^c\phi^d + \dots\right)\partial\phi^a\partial\phi^b$$

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Loop Amplitude:
($4^{pt.}$ loop level)

$$\mathscr{A}^{1-loop}_{abcd} \sim \int_k \left(f_s\,R_{a(nm)b} + f_a\,R_{a[nm]b}\right)\left(g_s\,R_{c(nm)d} + g_a\,R_{c[nm]d}\right)$$

Go to 1-Loop

- General Structure of the 1-Loop Amplitude

$$\mathscr{A}^{1-loop}_{abcd} \sim \int_k \left(f_s R_{a(nm)b} + f_a R_{a[nm]b}\right)\left(g_s R_{c(nm)d} + g_a R_{c[nm]d}\right)$$

Go to 1-Loop

- General Structure of the 1-Loop Amplitude in NLSM

$$\mathscr{A}^{1-loop}_{abcd} \sim \int_k \left(f_s\, R_{a(nm)b} R_{c(nm)d} + f_a\, R_{a[nm]b} R_{c[nm]d} \right)$$

- Counter term drops us out of nice picture

counter-term

$$\mathscr{L} = \frac{1}{2} g_{ab}(\phi)\, \partial\phi^a\, \partial\phi^b + \#\left(\partial\phi\right)^4$$

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?

Geometry - Kinematics Duality

Cheung, Helset, Parra-Martinez '22

- Inspired by Colour - Kinematics duality
- Allows for arbitrary field redefinitions

$$\phi^a \to \tilde{\phi}^a \left(\phi, \partial\phi, \ldots\ldots\right)$$

- Can handle theories of arbitrary massless bosons

$$\mathscr{L} \supset V(\phi), \partial\partial\phi, \ldots \qquad \& \qquad \phi, A_\mu, g_{\mu\nu}$$

Geometry - Kinematics Duality

Cheung, Helset, Parra-Martinez ’22

$$\mathcal{A}_4^{NLSM} = R_{abcd}\,t + R_{acbd}\,s$$

Geometry - Kinematics Duality

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$$\mathscr{A}_4^{NLSM} = R_{abcd}\, t + R_{acbd}\, s$$

replace

$$\mathscr{A}_4 = R\left(p_1^a, p_2^b, p_3^c, p_4^d\right)\, t + R\left(p_1^a, p_3^c, p_2^b, p_4^d\right)\, s$$

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- Duality Map

$$g_{ab} \to g_{ab}(p_1, p_2)\delta(p_1 + p_2)$$

Examples

- Clean Recipe:
  - Take NLSM Amplitude
  - Replace Geometry → Kinematics

$$\mathcal{A}_4^{NLSM} = R_{abcd}\, t + R_{acbd}\, s$$

$$\mathcal{A}_4 = R\left(p_1^a, p_2^b, p_3^c, p_4^d\right)\, t + R\left(p_1^a, p_3^c, p_2^b, p_4^d\right)\, s$$

$\phi^3 + \phi^4$ - Theory: $$R = g^2\left(\frac{1}{t^2} - \frac{1}{u^2}\right) + \frac{\lambda}{3}\left(\frac{1}{t} - \frac{1}{u}\right)$$

Nambu-Goldstone Theory: $$R \sim \lambda\,(t - u)$$

Generic Form: $$R \sim t^\rho - u^\rho$$

Absorb Counter - Term

$$\mathscr{L} = \frac{1}{2} g_{ab}(\phi)\, \partial\phi^a\, \partial\phi^b + \#\big(\partial\phi\big)^4$$

Geo-Kin Duality

Absorb Counter - Term

$$\mathscr{L} = \frac{1}{2} g_{ab}(\phi)\, \partial\phi^a\, \partial\phi^b + \#\big(\partial\phi\big)^4$$

Geo-Kin Duality

- Via the duality replacement: $g_{ab} \to g_{ab}(p_1, p_2)\delta(p_1 + p_2)$

$$g_{ab,cd} \to g_{ab|cd}(p_1, p_2 \,|\, p_3, p_4)\delta(p_1 + \cdots + p_4)$$

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- Compute kinematic Curvature

$$R_{abcd}(p_i) = \frac{1}{2}\left(g_{adbc}(p_1, p_4 \,|\, p_2, p_3) - g_{acbd}(p_1, p_3 \,|\, p_2, p_4) + \text{symmetrised}\right)$$

Curvature Renormalisation - NLSM

- Add counter term

$$\mathscr{L} = \frac{1}{2}\left(\delta_{ab} + \frac{1}{3}R_{acdb}\,\phi^c\phi^d + \frac{1}{4}\hat{g}_{abcd}\,\partial\phi^c\partial\phi^d\right)\partial\phi^a\partial\phi^b$$

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- Absorb into momentum dependent curvature tensor

$$\bar{R}_{abcd} = R_{abcd} + (p_a\cdot p_d + p_b\cdot p_c)\,\hat{g}_{adbc} - (p_a\cdot p_c + p_b\cdot p_d)\,\hat{g}_{acbd}$$

Curvature Renormalisation - NLSM

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- Solve Callan-Symanzik Equations $$\mu\frac{d\hat{g}_{abcd}}{d\mu} = 2\epsilon\hat{g}_{abcd}\,g_{ij,kl}\frac{\partial Z_{abcd}}{\partial g_{ij,kl}}$$
  - $\hat{g} \to \hat{g}(\mu)$

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  - $\hat{g} \to \hat{g}(\mu)$
- RG-improved Riemann Tensor $R\left(p_1^a, p_2^b, p_3^c, p_4^d; \mu\right)$

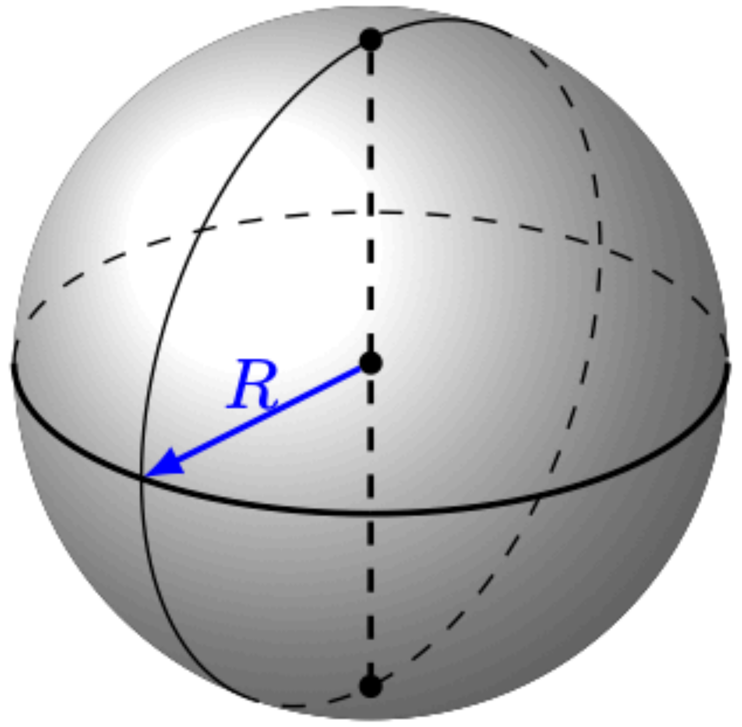
Curvature is now momentum and scale dependent; still obeys symmetries

Running Curvature - NLSM

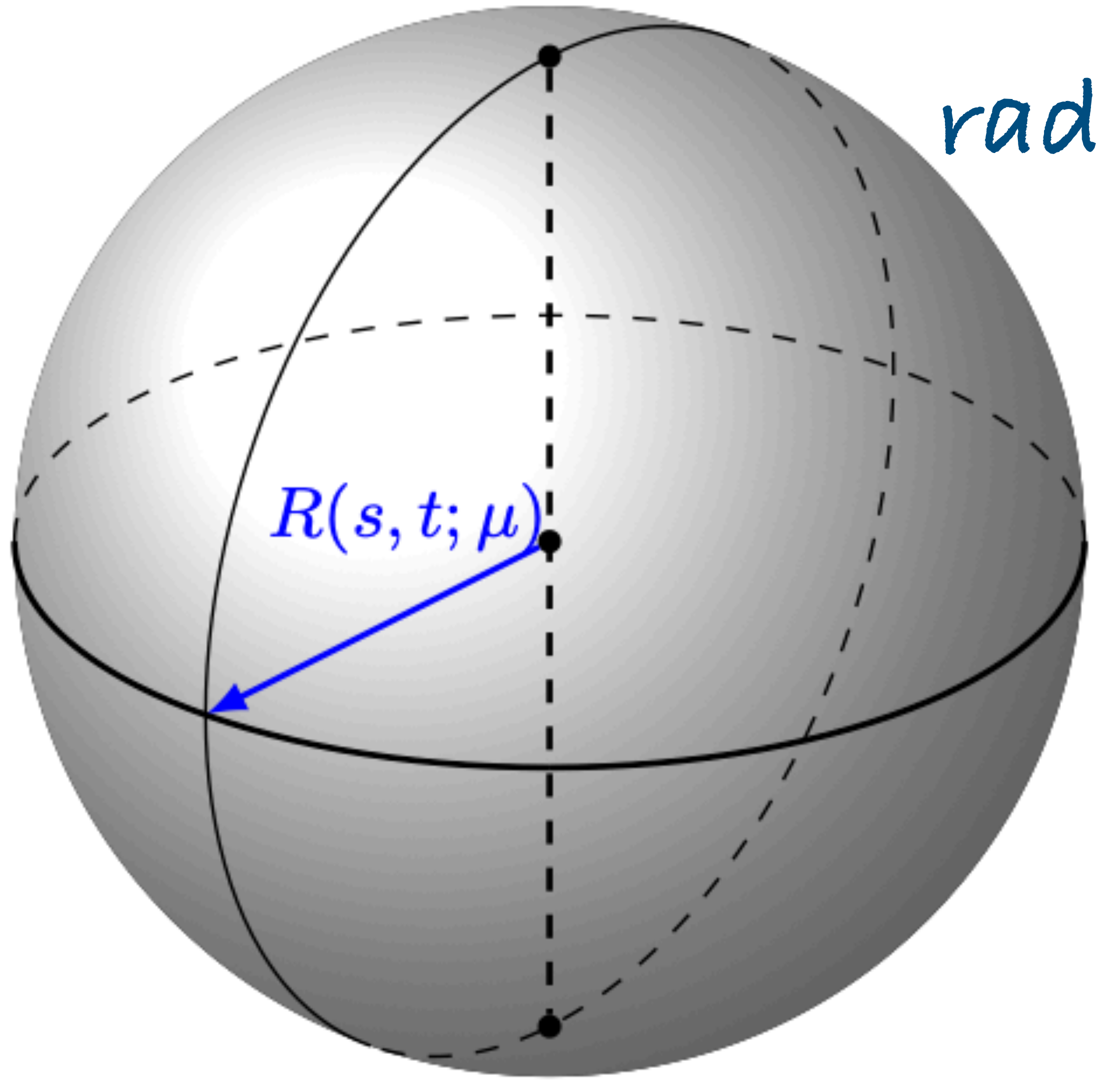
$$R(\mu) = \underbrace{\lambda N(N-1)}_{\mathcal{S}^n \text{ curvature}}\left(1 + \frac{\lambda}{6\pi^2} N \left(\underbrace{\frac{(N+2)}{(N-1)}(t-u)}_{\text{Momentum dependence}} - \underbrace{\log(\mu)\Big((3N-4)t - 5u\Big)}_{\text{running curvature}} \right)\right)$$

radius $= \lambda^{-\frac{1}{2}}$

tree-level



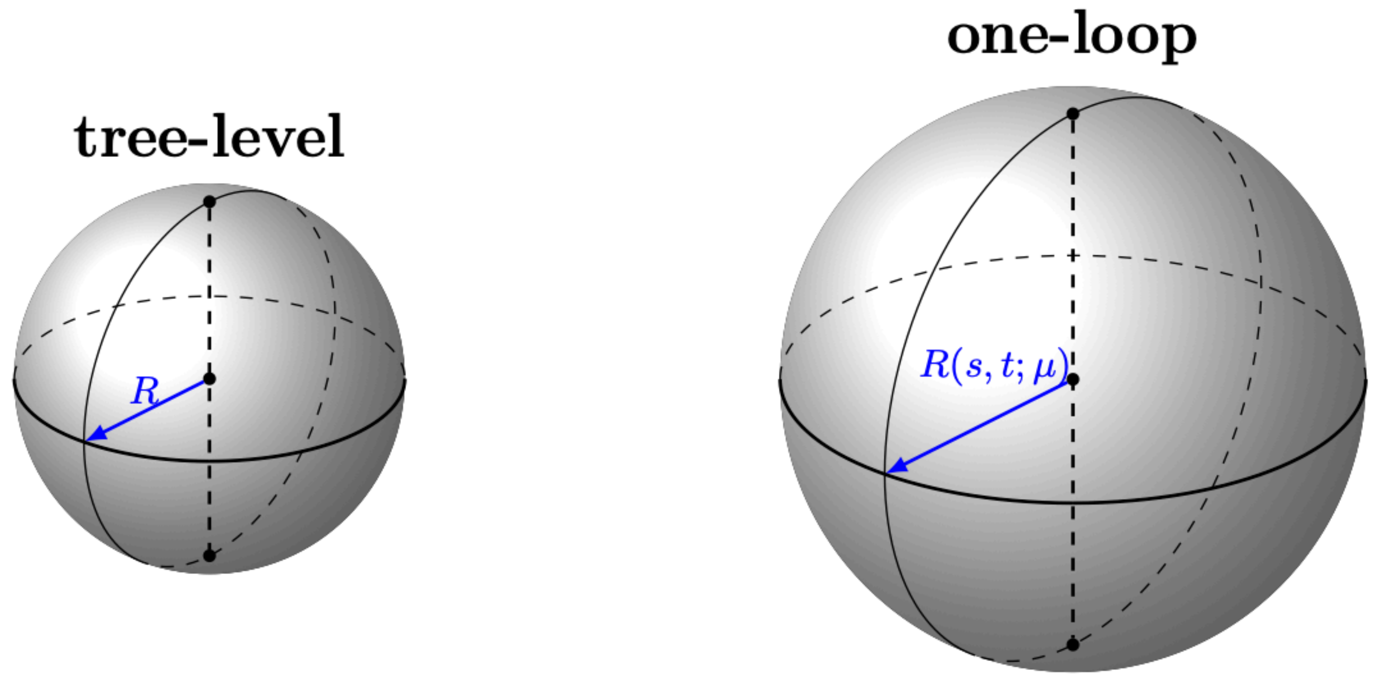
one-loop



radius $= \dfrac{1}{(\lambda f(p, \log \mu))^{\frac{1}{2}}}$

Conclusion

- Field-Space Manifold determines interactions
- Physical Observeables $\Longleftrightarrow$ Geometric Invariants
- Geometry-Kinematics duality generalises to arbitrary boson theories
- Universal structure of the curvature tensor at 1-loop
- "Geometric" picture of RG running
    - Loops induce running in curvature



$$R \longrightarrow R(s,t;\mu)$$