# Induced gravitational waves as cosmic tracers of leptogenesis

2504.20135 - M. Chianese, G. Domenech, T. Papanikolaou, S. Rome, N. Saviano

#### **Theodoros Papanikolaou**

Gravitational Wave Probes of Physics Beyond Standard Model 4, Faculty of Physics, University of Warsaw, 25/06/2025



Scuola Superiore Meridionale

• Leptogenesis is a 2-step process [M. Fukugita & T. Yanagida - 1986]

• Leptogenesis is a 2-step process [M. Fukugita & T. Yanagida - 1986]



• Leptogenesis is a 2-step process [M. Fukugita & T. Yanagida - 1986]



• Lepton Asymmetry through the Seesaw Mechanism

[Yanagida - 1979, Glashow - 1979, Gell-Mann et al. - 1980, Mohapatra & Senjanovic - 1980]

• Leptogenesis is a 2-step process [M. Fukugita & T. Yanagida - 1986]



• Lepton Asymmetry through the Seesaw Mechanism

[Yanagida - 1979, Glashow - 1979, Gell-Mann et al. - 1980, Mohapatra & Senjanovic - 1980]

• Lepton asymmetry is produced at  $T \sim M_N$ , where  $M_N \in [MeV, 10^{15} GeV]$ .

• Leptogenesis is a 2-step process [M. Fukugita & T. Yanagida - 1986]



• Lepton Asymmetry through the Seesaw Mechanism

[Yanagida - 1979, Glashow - 1979, Gell-Mann et al. - 1980, Mohapatra & Senjanovic - 1980]

- Lepton asymmetry is produced at  $T \sim M_N$ , where  $M_N \in [MeV, 10^{15} GeV]$ .
- Scales  $M_N > O(\text{TeV})$  are not reachable by colliders.

**Distinct Flavor Regimes of High-Scale Leptonesis:**  $|L_i \rangle = A_{i\alpha} |L_\alpha \rangle$ , where  $\alpha = e, \mu, \tau$ 

M <sub>N</sub> <109 GeV	10 <sup>9</sup> < M <sub>N</sub> <10 <sup>12</sup> GeV	>1012 GeV		
Three-flavor (3FL)	Two-flavor (2FL)	<b>One-flavor (1FL)</b> /vanilla regime	M <sub>N</sub>	
the rates of processes mediated by the $\tau$ and $\mu$ Yukawa coupling are non negligible—> effects of lepton flavors must be taken into account		no charged lepton flavour effects—> no sensitivity to the neutrino mixing matrix: mixing angles and low-energy CP phases.		

While the 1FL regime is not sensitive to neutrino mixing parameters, the 2FL and 3FL regimes can offer low-energy neutrino phenomenology, which however may not be unequivocal, unless the flavor structure of the theory is constrained by invoking symmetries.

cannot be tested by terrestrial experiments !!

It is therefore necessary to find other evidence to test the neutrino sector.

[Credits to N. Saviano]

• UV realisation of Type I seesaw mechanism based on  $U(1)_{B-L}$  symmetry with gauge coupling g', naturally embedded to GUTs.

$$-\Delta \mathcal{L} \subset \frac{1}{2} y_N \overline{N_R} \Phi N_R^C + y_D \overline{N_R} \tilde{H}^{\dagger} L + \frac{1}{4} \lambda_{H\Phi} H^2 \Phi^2 + V(\Phi, T)$$

• UV realisation of Type I seesaw mechanism based on  $U(1)_{B-L}$  symmetry with coupling g', naturally embedded to GUTs.

$$-\Delta \mathcal{L} \subset \frac{1}{2} y_N \overline{N_R} \Phi N_R^C + y_D \overline{N_R} \tilde{H}^{\dagger} L + \frac{1}{4} \lambda_{H\Phi} H^2 \Phi^2 + V(\Phi, T)$$

$$M_N, m_{\nu_i}$$

• UV realisation of Type I seesaw mechanism based on  $U(1)_{B-L}$  symmetry with gauge coupling g', naturally embedded to GUTs.

$$\begin{split} -\Delta \mathscr{L} \subset \frac{1}{2} y_N \overline{N_R} \Phi N_R^C + y_D \overline{N_R} \tilde{H}^{\dagger} L + \frac{1}{4} \lambda_{H\Phi} H^2 \Phi^2 + V(\Phi, T) \\ M_N, m_{\nu_i} \end{split} \qquad \qquad \text{Decay of } \Phi \end{split}$$

## **Our leptogenesis scenario** $-\Delta \mathscr{L} \subset \frac{1}{2} y_N \overline{N_R} \Phi N_R^C + y_D \overline{N_R} \tilde{H}^{\dagger} L + \frac{1}{4} \lambda_{H\Phi} H^2 \Phi^2 + V(\Phi, T)$



• UV realisation of Type-I seesaw mechanism based on  $U(1)_{B-L}$  symmetry with coupling g', naturally embedded to GUTs.

• At that time, right-handed neutrinos (RHNs) become massive with  $M_N = y_N v_{\Phi}$ , decaying later CP-asymmetrically into lepton doublets and Higgs bosons and producing the lepton asymmetry around  $T_{\rm lepto} \sim M_N$ .



• UV realisation of Type-I seesaw mechanism based on  $U(1)_{B-L}$  symmetry with coupling g', naturally embedded to GUTs.

• Later at  $T_{\rm dom} < M_N$ ,  $\Phi$  dominates triggering an early matter-dominated (eMD) era.

• UV realisation of Type-I seesaw mechanism based on  $U(1)_{B-L}$  symmetry with coupling g', naturally embedded to GUTs.

$$-\Delta \mathscr{L} \subset \frac{1}{2} y_N \overline{N_R} \Phi N_R^C + y_D \overline{N_R} \tilde{H}^{\dagger} L + \frac{1}{4} \lambda_{H\Phi} H^2 \Phi^2 + V(\Phi, T)$$
  
$$M_N, m_{\nu_i} \qquad \qquad \text{Decay of } \Phi$$

- Later at  $T_{\rm dom} < M_N$ ,  $\Phi$  dominates triggering an early matter-dominated (eMD) era.
- Once the SM Higgs takes its vev  $v_h = 174 \text{GeV}$  at the electroweak (EW) phase transition, light neutrinos become massive  $m_{\nu_i} \sim y_D^2 v_h^2 / M_N$  via the Type-I seesaw mechanism.

• UV realisation of Type-I seesaw mechanism based on  $U(1)_{B-L}$  symmetry with coupling g', naturally embedded to GUTs.

$$-\Delta \mathscr{L} \subset \frac{1}{2} y_N \overline{N_R} \Phi N_R^C + y_D \overline{N_R} \tilde{H}^{\dagger} L + \frac{1}{4} \lambda_{H\Phi} H^2 \Phi^2 + V(\Phi, T)$$
  
$$M_N, m_{\nu_i} \qquad \qquad \text{Decay of } \Phi$$

- Later at  $T_{\rm dom} < M_N$ ,  $\Phi$  dominates triggering an early matter-dominated (eMD) era.
- Once the SM Higgs takes its vev  $v_h = 174 \text{GeV}$  at the electroweak (EW) phase transition, light neutrinos become massive  $m_{\nu_i} \sim y_D^2 v_h^2 / M_N$  via the Type-I seesaw mechanism.
- Finally, the early matter-dominated (eMD) era triggered by  $\Phi$  ends through the decay channel  $\Phi \rightarrow hh$ .

## Long lived $\Phi$

• In order to make  $\Phi$  long lived one should require that  $\Phi\to Z'Z'$  and  $\Phi\to NN\,$  are kinematically forbidden leading to

• In order to make  $\Phi$  long lived one should require that  $\Phi \to Z'Z'$  and  $\Phi \to NN$  are kinematically forbidden leading to

$$M_{Z'} = \sqrt{2}g' v_{\Phi} > m_{\Phi} \Rightarrow g' < 1 \quad \text{and} \quad M_N = y_N v_{\Phi} > m_{\Phi} = \sqrt{2\lambda} v_{\Phi} \Rightarrow y_N > \sqrt{2g'^3} \,.$$

• In order to make  $\Phi$  long lived one should require that  $\Phi \to Z'Z'$  and  $\Phi \to NN$  are kinematically forbidden leading to

$$M_{Z'} = \sqrt{2}g' v_{\Phi} > m_{\Phi} \Rightarrow g' < 1 \quad \text{and} \quad M_N = y_N v_{\Phi} > m_{\Phi} = \sqrt{2\lambda} v_{\Phi} \Rightarrow y_N > \sqrt{2g'^3} \,.$$

• One should also account for the competitive channel  $\Phi \to f\bar{f}V$  requiring that  $\Gamma_{\Phi}^{hh}(M_N, y_N, g') \ge \Gamma_{\Phi}^{f\bar{f}V}(M_N, y_N, g')$ .

- In order to make  $\Phi$  long lived one should require that  $\Phi \to Z'Z'$  and  $\Phi \to NN$  are kinematically forbidden leading to  $M_{Z'} = \sqrt{2}g'v_{\Phi} > m_{\Phi} \Rightarrow g' < 1$  and  $M_N = y_N v_{\Phi} > m_{\Phi} = \sqrt{2\lambda}v_{\Phi} \Rightarrow y_N > \sqrt{2g'^3}$ .
- One should also account for the competitive channel  $\Phi \to f\bar{f}V$  requiring that  $\Gamma_{\Phi}^{hh}(M_N, y_N, g') \ge \Gamma_{\Phi}^{f\bar{f}V}(M_N, y_N, g')$ .
- Finally, one should require the lepton asymmetry takes place at  $T_{\text{lepto}} \sim M_N < T_c = 2\sqrt{g'}v_{\Phi} \Rightarrow y_N < 2\sqrt{g'}$ .

## The overall picture





## What is the GW phenomenology of such an eMD era?

• Primordial induced GWs are generated through second order gravitational effects:  $\mathscr{L}_{\Psi,h}^{(3)} \ni h\Psi^2$  [S. Matarrese et al. - 1993, 1994, 1998, G. Domenech - 2021].

• Primordial induced GWs are generated through second order gravitational effects:  $\mathscr{L}_{\Psi,h}^{(3)} \ni h\Psi^2$  [S. Matarrese et al. - 1993, 1994, 1998, G. Domenech - 2021].

$$\mathscr{P}_{h}(\eta,k) \propto \int \mathrm{d}v \int \mathrm{d}u \left( \int f(v,u,k,\eta) \mathrm{d}\eta \right)^{2} \mathscr{P}_{\mathscr{R}}(kv) \mathscr{P}_{\mathscr{R}}(ku) \,.$$

• Primordial induced GWs are generated through second order gravitational effects:  $\mathscr{L}_{\Psi,h}^{(3)} \ni h\Psi^2$  [S. Matarrese et al. - 1993, 1994, 1998, G. Domenech - 2021].

$$\mathcal{P}_{h}(\eta,k) \propto \int \mathrm{d}v \int \mathrm{d}u \left( \int f(v,u,k,\eta) \mathrm{d}\eta \right)^{2} \mathcal{P}_{\mathcal{R}}(kv) \mathcal{P}_{\mathcal{R}}(ku) \,.$$
$$\mathcal{Q}_{\mathrm{GW}}(\eta,k) \equiv \frac{1}{\rho_{\mathrm{tot}}} \frac{\mathrm{d}\rho_{\mathrm{GW}}}{\mathrm{d}\ln k} = \frac{1}{24} \left( \frac{k}{a(\eta)H(\eta)} \right)^{2} \mathcal{P}_{h}(\eta,k),$$

• Primordial induced GWs are generated through second order gravitational effects:  $\mathscr{L}_{\Psi,h}^{(3)} \ni h\Psi^2$  [S. Matarrese et al. - 1993, 1994, 1998, G. Domenech - 2021].

$$\mathscr{P}_{h}(\eta,k) \propto \int \mathrm{d}v \int \mathrm{d}u \left( \int f(v,u,k,\eta) \mathrm{d}\eta \right)^{2} \mathscr{P}_{\mathscr{R}}(kv) \mathscr{P}_{\mathscr{R}}(ku) \,.$$
$$\mathscr{Q}_{\mathrm{GW}}(\eta,k) \equiv \frac{1}{\rho_{\mathrm{tot}}} \frac{\mathrm{d}\rho_{\mathrm{GW}}}{\mathrm{d}\ln k} = \frac{1}{24} \left( \frac{k}{a(\eta)H(\eta)} \right)^{2} \mathscr{P}_{h}(\eta,k),$$

• We will focus in our scenario on **GWs induced** during the  $\Phi$ -driven eMD era.

• During an eMD phase, sub-horizon density perturbations grow, i.e.  $\delta_k \propto a$ , as well as velocity flows. One then expects structure ( $\Phi$  halo) formation in the non-linear regime [Barenboim & Rasero - 2013].

- During an eMD phase, sub-horizon density perturbations grow, i.e.  $\delta_k \propto a$ , as well as velocity flows. One then expects structure ( $\Phi$  halo) formation in the non-linear regime [Barenboim & Rasero 2013].
- The scale of non-linearities at a given conformal time  $\eta$  is estimated as [Assadullahi & Wands 2009, Kohri & Terada 2018]

$$k_{\rm NL}(\eta) \sim \alpha A_{\rm s}^{-1/4} \mathcal{H}(\eta)$$
.

where we assumed a scale-invariant curvature power spectrum with amplitude  $A_{\rm s}$ , and  $\alpha \simeq 1.7$ . Modes with  $k > k_{\rm NL}$  are in the non-linear regime, i.e.  $\mathscr{P}_{\delta}(k > k_{\rm NL}) > 1$ .

- During an eMD phase, sub-horizon density perturbations grow, i.e.  $\delta_k \propto a$ , as well as velocity flows. One then expects structure ( $\Phi$  halo) formation in the non-linear regime [Barenboim & Rasero 2013].
- The scale of non-linearities at a given conformal time  $\eta$  is estimated as [Assadullahi & Wands 2009, Kohri & Terada 2018]

$$k_{\rm NL}(\eta) \sim \alpha A_{\rm s}^{-1/4} \mathcal{H}(\eta) \, . \label{eq:knl}$$

where we assumed a scale-invariant curvature power spectrum with amplitude  $A_s$  and  $\alpha \simeq 1.7$ . Modes with  $k > k_{\rm NL}$  are in the non-linear regime, i.e.  $\mathcal{P}_{\delta}(k > k_{\rm NL}) > 1$ .

• IGWs on linear scales  $k < k_{\rm NL}$  are negligible. IGW generation is abundant on non-linear scales and is dominated by the largest structures that form at the end of the eMD era [Fernandez et al. - 2024].

• We need thus to require that at least the largest possible non-linear scale enters the Hubble radius during the eMD, i.e.  $k_{\rm NL}(\eta_{\rm dec}) < \mathcal{H}_{\rm dom}$ .

• We need thus to require that at least the largest possible non-linear scale enters the Hubble radius during the eMD, i.e.  $k_{\rm NL}(\eta_{\rm dec}) < \mathcal{H}_{\rm dom}$ .

$$A_{\rm s} \ge A_{\rm s,min} = \alpha^4 \left(\frac{a_{\rm dom}}{a_{\rm dec}}\right)^2 \simeq 9 \left(\frac{T_{\rm dec}}{T_{\rm dom}}\right)^2,$$

• We need thus to require that at least the largest possible non-linear scale enters the Hubble radius during the eMD, i.e.  $k_{\rm NL}(\eta_{\rm dec}) < \mathcal{H}_{\rm dom}$ .

$$A_{\rm s} \ge A_{\rm s,min} = \alpha^4 \left(\frac{a_{\rm dom}}{a_{\rm dec}}\right)^2 \simeq 9 \left(\frac{T_{\rm dec}}{T_{\rm dom}}\right)^2,$$

with 
$$T_{\text{dom}} = \frac{\rho_{\Phi}(T_c)}{\rho_{\text{r}}(T_c)} T_c$$
 and  $T_{\text{dec}} = \tilde{T}_{\text{dec}} \ln\left(\frac{\Lambda_{\text{GUT}}}{\mu}\right)$ ,

where 
$$\tilde{T}_{dec} = \left(\frac{72}{5\pi^2 g_*}\right)^{1/4} 10^{-2} \left[ M_{Pl} \Gamma_0 \frac{y_N^3}{\sqrt{g'^3}} \left(\frac{M_N}{10^{13} \text{GeV}}\right)^3 \right]^{1/2}$$
  
[Chianese et al. - 2024]

 Borrowing results of numerical simulations [Eggemeier et al. - 2023, Fernandez et al. - 2024, Dalianis & Kouvaris - 2024], the GW signal can be fitted on nonlinear scales quite well as

$$\Omega_{\rm GW}(k) \simeq 0.05 A_{\rm s}^{7/4} \left(\frac{k}{\mathcal{H}_{\rm dec}}\right)^{3/2} \text{ for } k_{\rm low} \le k \le k_{\rm high},$$
  
where  $k_{\rm low} = 15\mathcal{H}_{\rm dec}$  and  $k_{\rm high} = 9k_{\rm NL} = 14\mathcal{H}_{\rm dec}/A_{\rm s}^{1/4}.$ 

Borrowing results of numerical simulations [Eggemeier et al. - 2023, Fernandez et al. - 2024, Dalianis & Kouvaris - 2024], the GW signal can be fitted on non-linear scales quite well as

$$\Omega_{\rm GW}(k) \simeq 0.05 A_{\rm s}^{7/4} \left(\frac{k}{\mathcal{H}_{\rm dec}}\right)^{3/2} \text{ for } k_{\rm low} \le k \le k_{\rm high},$$
  
where  $k_{\rm low} = 15\mathcal{H}_{\rm dec}$  and  $k_{\rm high} = 9k_{\rm NL} = 14\mathcal{H}_{\rm dec}/A_{\rm s}^{1/4}.$ 

• Analytical studies considering GW induced by early structure pancake gravitational collapse [Dalianis & Kouvaris - 2021, Flores et al. - 2023] suggest that for  $k > k_{\rm high}$ ,  $\Omega_{\rm GW} \propto 1/k^n$  with  $n \ge 1$ .

Borrowing results of numerical simulations [Eggemeier et al. - 2023, Fernandez et al. - 2024, Dalianis & Kouvaris - 2024], the GW signal can be fitted on non-linear scales quite well as

$$\Omega_{\rm GW}(k) \simeq 0.05 A_{\rm s}^{7/4} \left(\frac{k}{\mathcal{H}_{\rm dec}}\right)^{3/2} \quad \text{for} \quad k_{\rm low} \le k \le k_{\rm high},$$
  
where  $k_{\rm low} = 15\mathcal{H}_{\rm dec}$  and  $k_{\rm high} = 9k_{\rm NL} = 14\mathcal{H}_{\rm dec}/A_{\rm s}^{1/4}.$ 

• Analytical studies considering GW induced by early structure pancake gravitational collapse [Dalianis & Kouvaris - 2021, Flores et al. - 2023] suggest that for  $k > k_{\rm high}$ ,  $\Omega_{\rm GW} \propto 1/k^n$  with  $n \ge 1$ .

$$f_{\rm high} \equiv \frac{k_{\rm high}}{2\pi} = \frac{14\mathcal{H}_{\rm dec}A_{\rm s}^{-1/4}}{2\pi} \propto \frac{M_N^3 y_N^3}{\sqrt{g'^3}} A_s^{-1/4}$$

• One can establish a remarkable connection between leptogenesis and IGWs since  $\Gamma_{\Phi}^{hh}(y_N, M_N, g') = \mathcal{H}_{dec}/a_{dec}$ .

- Working with frequencies f instead of wavenumbers k, one can recast  $\Omega_{\rm GW}(k)$  as

$$\Omega_{\rm GW,0} h^2 = 4.2 \times 10^{-5} A_s^{11/8} \left(\frac{f}{f_{\rm high}}\right)^{3/2},$$
  
where  $f_{\rm high} \simeq 6.4 \times 10^{-5} \text{Hz} \left(\frac{A_s}{10^{-5}}\right)^{-1/4} \left(\frac{T_{\rm dec}}{10 \text{ GeV}}\right)$  and  
 $f_{\rm low} \simeq 3.8 \times 10^{-6} \text{Hz} \left(\frac{T_{\rm dec}}{10 \text{ GeV}}\right)$ 

## Results



#### **Benchmark parameter values**

	g'	$M_N \; [\text{GeV}]$	$y_N$		$A_s$
BP1	$10^{-2}$	$1.4 \times 10^{8}$	$1.4 \times 10^{-3}$	1	$.18 \times 10^{-5}$
BP2	$10^{-2}$	$2.8 \times 10^{11}$	$1.4 \times 10^{-3}$	9	$.96 \times 10^{-3}$
BP3	$10^{-2}$	$1.5 \times 10^6$	$1.8 \times 10^{-2}$	1	$.00 \times 10^{-2}$
BP4	$10^{-2}$	$1.0 \times 10^{9}$	$2.0 \times 10^{-3}$	4	$.30 \times 10^{-4}$
BP5	$10^{-3}$	$1.0 \times 10^{6}$	$1.5 \times 10^{-4}$	3	$.00 \times 10^{-3}$

#### **Benchmark parameter values**



#### Detectability by GW observatories



• There is a direct link between leptogenesis and the IGW spectrum. In particular,  $f_{\rm high}$  is directly related to  $M_N$ .

- There is a direct link between leptogenesis and the IGW spectrum. In particular,  $f_{\rm high}$  is directly related to  $M_N$ .
- Different flavor regimes, characterized by different  $M_N$  values produce GWs at varying frequencies.

- There is a direct link between leptogenesis and the IGW spectrum. In particular,  $f_{\rm high}$  is directly related to  $M_N$ .
- Different flavor regimes, characterized by different  $M_N$  values produce GWs at varying frequencies.

**Future Perspectives** 

- There is a direct link between leptogenesis and the IGW spectrum. In particular,  $f_{\rm high}$  is directly related to  $M_N$ .
- Different flavor regimes, characterized by different  $M_N$  values produce GWs at varying frequencies.

#### **Future Perspectives**

• eMD eras favor the formation of PBHs. Even with  $A_{\rm s} \sim 10^{-4}$ , one can produce substantial PBH abundances [Harada et al - 2016, Ballesteros et al. - 2019].

- There is a direct link between leptogenesis and the IGW spectrum. In particular,  $f_{\rm high}$  is directly related to  $M_N$ .
- Different flavor regimes, characterized by different  $M_N$  values produce GWs at varying frequencies.

#### **Future Perspectives**

- **eMD eras favor** the formation of PBHs. Even with  $A_{\rm s} \sim 10^{-4}$ , one can produce **substantial PBH abundances** [Harada et al 2016, Ballesteros et al. 2019].
- Any BSM physics framework involving an eMD epoch triggered by long lived particles can be readily explored through IGWs.

## Thanks for your attention

## Appendix

## **Quasi-Degeneracy of RHNs**

$$M_N > T_{\rm dom} \Rightarrow \Delta \simeq T_{\rm dec}/T_{\rm dom} \Rightarrow \eta_{\rm B} \simeq 10^{-3} \left( \frac{3m_\nu M_{N_i}}{8\pi v_h^2 \delta} \right) \Delta$$

where  $\delta = \frac{M_{N_i} - M_{N_j}}{M_{N_i}}$  is the quasi – degenarcy among RHNs.

$$\eta_B \simeq 6.3 \times 10^{-10} \Rightarrow \delta \simeq 0.2 \left(\frac{M_{N_i}}{10^9 \text{GeV}}\right) \sqrt{A_s^{\text{min}}}$$

 Strong (weak) amplitude GWs are associated with weakly (strongly) quasidegenerate RHNs.

#### **Basics of Scalar Induced Gravitational Waves**

• Choosing as the gauge for the GW frame the Newtonian gauge, the metric is written as  $\begin{bmatrix} h_{ii} \end{bmatrix} = \frac{1}{2}$ 

$$ds^{2} = a^{2}(\eta) \left\{ -(1+2\Phi)d\eta^{2} + \left[ (1-2\Phi)\delta_{ij} + \frac{h_{ij}}{2} \right] dx^{i}dx^{j} \right\}.$$

• The equation of motion for the Fourier modes,  $h_{\vec{k}}$ , read as:

$$h_{\vec{k}}^{s,"} + 2\mathcal{H}h_{\vec{k}}^{s,'} + k^2 h_{\vec{k}}^s = 4S_{\vec{k}}^s.$$

• The source term,  $S_{\vec{k}}$  can be recast as:

$$S_{\vec{k}}^{s} = \int \frac{\mathrm{d}^{3}\vec{q}}{(2\pi)^{3/2}} e_{ij}^{s}(\vec{k})q_{i}q_{j} \left[ 2\Phi_{\vec{q}}\Phi_{\vec{k}-\vec{q}} + \frac{4}{3(1+w)} (\mathscr{H}^{-1}\Phi_{\vec{q}}' + \Phi_{\vec{q}})(\mathscr{H}^{-1}\Phi_{\vec{k}-\vec{q}}' + \Phi_{\vec{k}-\vec{q}}) \right]$$

• The GW spectral abundance can be written as

$$\Omega_{\rm GW}(\eta,k) \equiv \frac{1}{\rho_{\rm tot}} \frac{\mathrm{d}\rho_{\rm GW}}{\mathrm{d}\ln k} = \frac{1}{24} \left(\frac{k}{a(\eta)H(\eta)}\right)^2 \mathscr{P}_h(\eta,k)$$

with 
$$\mathscr{P}_{h}(\eta, k) \equiv \frac{k^{3} |h_{k}|^{2}}{2\pi^{2}} \propto \int dv \int du \left( \int f(v, u, k, \eta) d\eta \right)^{2} \mathscr{P}_{\Phi}(kv) \mathscr{P}_{\Phi}(ku).$$