Cosmological phase transitions, nonperturbatively

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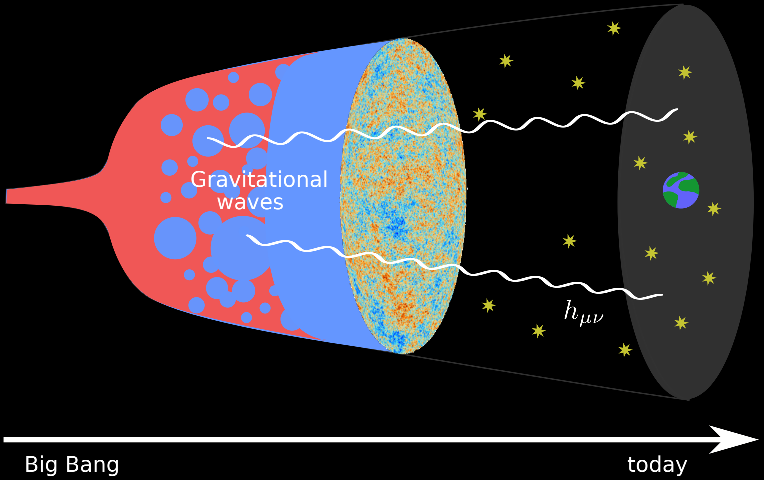
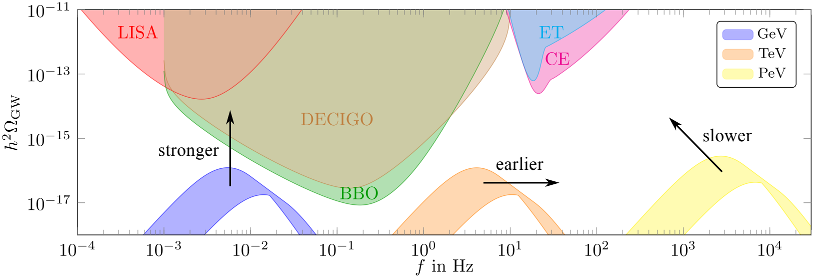
Gravitational Wave Probes of Physics Beyond Standard Model 4
25 June 2025


Gravitational waves
$h_{\mu\nu}$
Big Bang
today

Gravitational waves from phase transitions

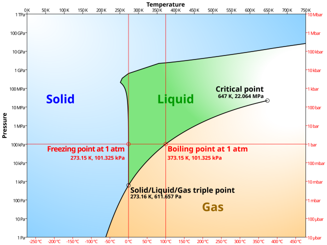


Confinement transitions in dark $\mathrm{SU}(6)$ Yang-Mills.

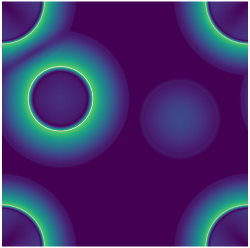
Huang et al. '21

Understanding phase transitions

$\mathscr{L}_{\text{microphysics}} \rightarrow$

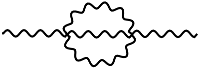


$\rightarrow$

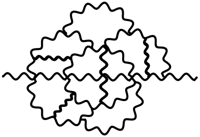


Why nonperturbative?

Even if g^2 is small, but the temperature is high



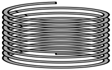
$$\approx \quad g^4 \int_{p_1 p_2} \frac{1}{p_i^6} n_B(p_i)^2 \approx g^4 T^2$$



$$\approx \quad g^{24} \int_{p_1 \ldots p_{12}} \frac{1}{p_i^{46}} n_B(p_i)^{12}$$

$$\approx \quad g^{24} \frac{(g^2 T)^{48}}{(g^2 T)^{46}} \left(\frac{1}{g^2}\right)^{12} \approx g^4 T^2$$

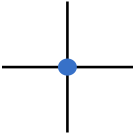
Linde '80



Lattice QFT

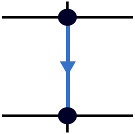
Review of lattice QFT: discretisation

- Scalars, fermions live on sites: $\phi(x)$, $\psi(x)$.



- Gauge fields parallel transport along links.

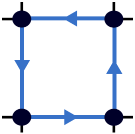
$$\underbrace{U_\mu(x)}_{\in\text{group}} = \exp\left[-i\int_x^{x+a\hat{\mu}} \underbrace{\mathcal{A}_\nu(x')}_{\in\text{algebra}} \mathrm{d}x'_\nu\right]$$



- Field strengths circle around plaquettes.

$$U_{\mu\nu}(x) = U_\mu(x)U_\nu(x+a\hat{\mu})U_\mu^\dagger(x+a\hat{\nu})U_\nu^\dagger(x)$$

$$\sum_{\mu\nu}\mathrm{Tr}(1-U_{\mu\nu}(x)) \to \frac{a^4}{4}\mathrm{Tr}F_{\mu\nu}F_{\mu\nu} + O(a^2)$$



Review of lattice QFT: sampling probabilities

- Wick rotate $t \to it_{\mathrm{E}}$ the whole path integral

$$\int \mathcal{D}\phi\mathcal{D}\psi\mathcal{D}A \, \underbrace{e^{iS[\phi,\psi,A]}}_{\text{phase}} \to \int \mathcal{D}\phi\mathcal{D}\psi\mathcal{D}A \, \underbrace{e^{-S_{\mathrm{E}}[\phi,\psi,A]}}_{\text{probability}}$$

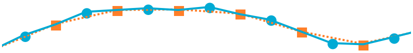
- Now tackle with Monte-Carlo importance sampling

$$\langle\phi(x)\rangle = \frac{1}{N_{\text{samples}}} \sum_{i=1}^{N_{\text{samples}}} \phi_i(x)$$

with $\phi_i(x)$ a random variable drawn from $P(\phi) \propto e^{-S_{\mathrm{E}}[\phi,\psi,A]}$.

- Errors decrease as $\propto 1/\sqrt{N_{\text{samples}}}$, independent of dimensionality.

Review of lattice QFT: fluctuating fields



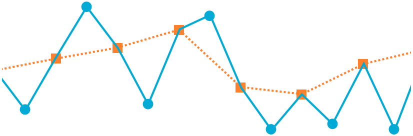
In classical field theory problems, fields are smooth on short scales.

$$\Rightarrow \left| \frac{\phi_{i+1} - \phi_i}{a} \right| = O(1)$$

But quantum fields are noisy!

$$\exp\left[-a^4 \left(\frac{\phi_{i+1} - \phi_i}{a} \right)^2 \right] = O(1)$$

$$\Rightarrow \left| \frac{\phi_{i+1} - \phi_i}{a} \right| = O\left(a^{-2}\right)$$



Review of lattice QFT: continuum limit

- Input lattice parameters are bare parameters, e.g.

$$\mathscr{L}_{\text{lat}} \supset \frac{1}{2} m_{\text{lat}}^2 \phi(x)^2 + \frac{1}{4!} \lambda_{\text{lat}} \phi(x)^4$$

where e.g. $m_{\text{lat}}^2 \propto a^{-2}$ and $\lambda_{\text{lat}} \propto \log a$ in the continuum limit.

- So must measure the physical mass, e.g. by fitting

$$\langle \phi(0)\phi(x) \rangle \propto \frac{e^{-m_{\text{phys}}|x|}}{|x|^2}$$

- Change bare input parameters to follow lines of constant physics as $a \to 0_+$.

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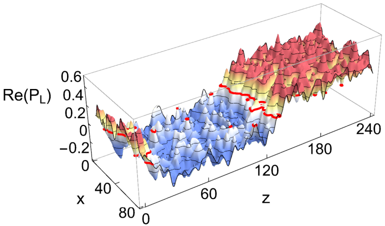
Review of lattice QFT complete!

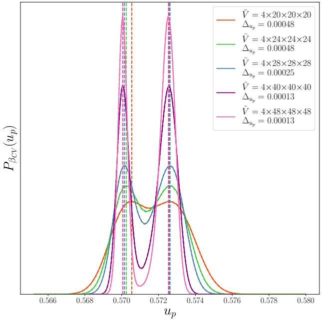
Recent lattice results

- First-order confinement transitions in $\mathrm{SU}(N)$, $\mathrm{Sp}(N)$
- Large N behaviour, e.g. $L/T_c^4 = 0.360(6)N^2 - 1.88(17)$
- Density of states method for strong transitions



SU(8) confinement interface

Rindlisbacher et al. '25



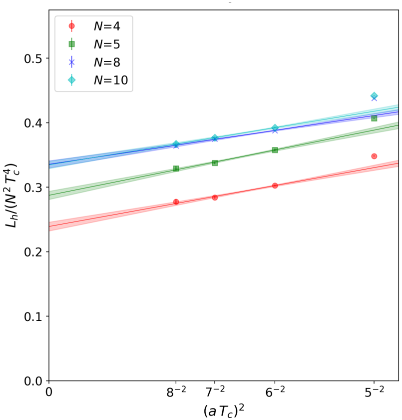
Sp(4) plaquette distribution

Bennett et al. '24

see also Springer et al. '23

Lattice Monte-Carlo strengths

- Strong couplings are easy
- Computable errors
  - Statistical errors often $\sim 1\%$
  - Systematic errors from continuum limit also often $\sim 1\%$



Continuum limit of latent heat

Rindlisbacher et al. '25

Lattice Monte-Carlo weaknesses

- Time dependent quantities are not computable, as the real-time sign problem prevents importance sampling,

$$\langle \mathcal{O}(t)\mathcal{O}(0)\rangle = \frac{1}{Z}\int \mathcal{D}\phi \; \mathcal{O}(t)\mathcal{O}(0)e^{iS[\phi]}$$

  and analytic continuation of noisy data is generally ill posed.

- Some models don't exist on the lattice, notably chiral non-Abelian gauge couplings, so the $\mathrm{SU}(2)_{\mathrm{w}}$ interactions of the SM can't be put on the lattice. Nielsen & Ninomiya '81

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We will see ways around these below!

Extending lattice methods with EFT

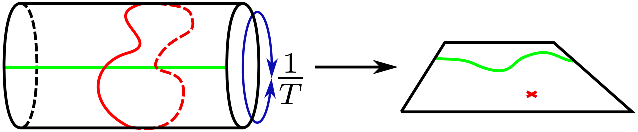
High temperature QFT

- Thermodynamics $Z = \mathrm{Tr} e^{-\hat{H}/T}$ formulated in $\mathbb{R}^3 \times S^1$

$$\Rightarrow \Psi(\tau, \mathbf{x}) = \sum_n \psi_n(\mathbf{x}) e^{i(n\pi T)\tau}$$

- These modes have masses $m_n^2 = m^2 + (n\pi T)^2$, so on energy scales $\ll \pi T$, the nonzero modes can be integrated out

$$\underbrace{\int_{\mathbb{R}^3 \times S^1} \mathscr{L}}_{\text{bosons and fermions}} \quad \rightarrow \quad \underbrace{\int_{\mathbb{R}^3} \mathscr{L}_{\text{eff}}}_{\text{just bosons}}$$



Kajantie et al. '95, Braaten & Nieto '95

3d effective theories

- In a 3d EFT, canonical mass dimensions are different

$$\int_{\mathbb{R}^3} \mathscr{L}_{\text{eff}} \supset \int_{\mathbb{R}^3} \Bigg[\underbrace{\frac{1}{2}\partial_i A_0^a \partial_i A_0^b}_{[A_0^a]=1/2} - \underbrace{\bar{g}_{abc} A_i^a A_0^b \partial^i A_0^c}_{[\bar{g}_{abc}^2]=1} + \frac{\bar{m}_{ab}^2}{2} A_0^a A_0^b \Bigg].$$

  Marginal interactions become relevant $\to$ superrenormalisable.
- Superrenormalisability $\Rightarrow$ continuum limit simple. Laine '95
- The loop expansion parameter changes

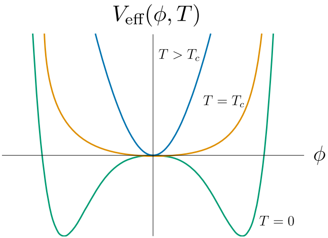
$$\alpha \equiv \frac{g^2}{(4\pi)^2} \to \alpha_3 \equiv \frac{g_3^2}{(4\pi) m_3},$$

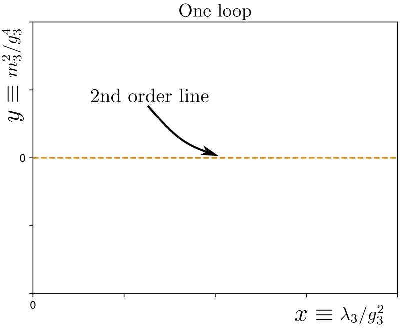
  and diverges for massless interacting fields. Linde '80

Electroweak phase diagram

One-loop

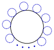
$$V_{\text{eff}} = V_0 + \bigcirc$$
$$= \frac{1}{2} m_3^2(T)\phi^2 + \frac{1}{4}\lambda_3(T)\phi^4$$



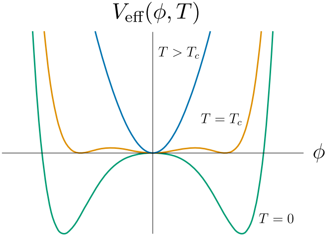


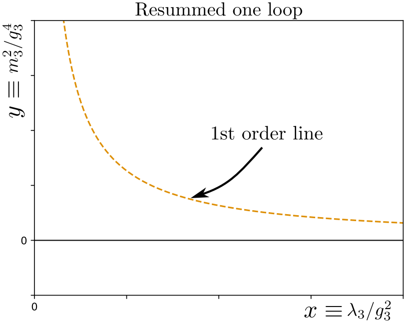
Electroweak phase diagram

Resummations

$V_{\text{eff}} = V_0 +$  $+$ 

$$= \frac{1}{2}m_3^2\phi^2 + \frac{1}{4}\lambda_3\phi^4 - \frac{g_3^3}{16\pi}(\phi^2)^{3/2}$$





An EFT expansion in $\sqrt{\lambda_3}/g_3$:

NLO — Arnold & Espinosa '92

N^2LO — Ekstedt, OG & Löfgren '22

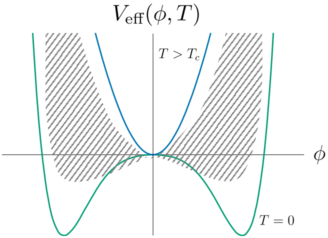
N^4LO — Ekstedt, Schicho & Tenkanen '24

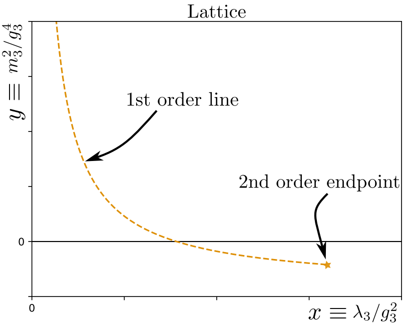
Electroweak phase diagram

Infrared breakdown

$$V_{\text{eff}} = ?$$

All higher loops $\subset$ N^{5}LO. Linde '80





Resolve by 3d lattice simulations.

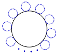
Farakos et al. '94, Kajantie et al. '96

Csikor, Fodor & Heitger '98

OG, Güyer & Rummukainen '22

Order of the EW phase transition? A potted history

- Leading order (LO): $V_T = V_0 +$ 

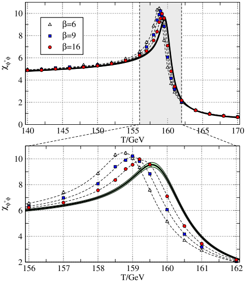
  $\Rightarrow$ 2$^{\text{nd}}$ order — Dolan & Jackiw '73
- NLO: $V_T = V_0 +$  $+$ 

  $\Rightarrow$ 1$^{\text{st}}$ order — Arnold & Espinosa '92
- Infrared problems at higher orders

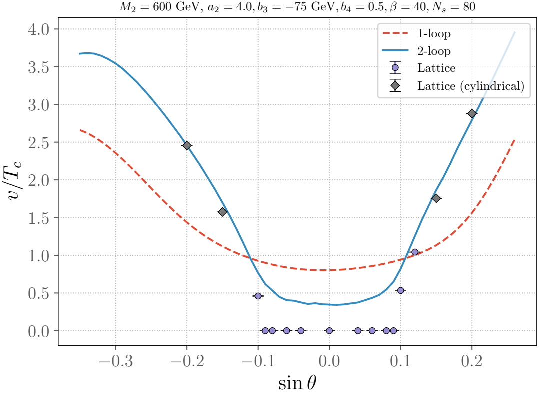
  $\Rightarrow$? order — Linde '80
- EFT + lattice approach resolves all issues

  $\Rightarrow$ crossover — Kajantie et al '96
- Accurate thermodynamics for SM

  D'Onofrio & Rummukainen '15, Laine & Meyer '15



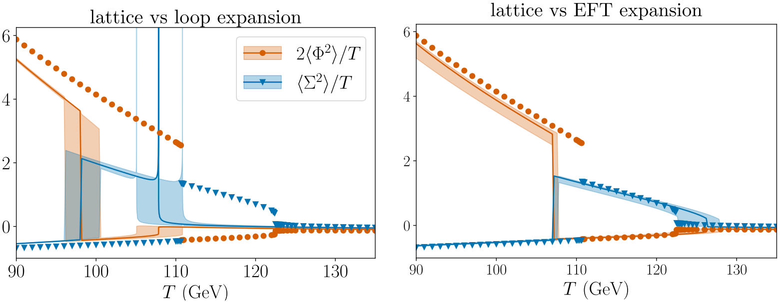
Lattice simulations of the xSM



$$\mathscr{L} = \mathscr{L}_{\mathrm{SM}} - \frac{a_1}{2}(\Phi^\dagger\Phi)S - \frac{a_2}{2}(\Phi^\dagger\Phi)S^2$$
$$- \frac{1}{2}(\partial S)^2 - \frac{m_S^2}{2}S^2 - \frac{b_3}{3}S^3 - \frac{b_4}{4}S^4$$

Niemi et al. '24

Testing perturbative approaches with the lattice



$$\mathscr{L} = \mathscr{L}_{\mathsf{SM}} - \frac{a_2}{2}\Phi^\dagger\Phi\Sigma^a\Sigma^a - \frac{1}{2}D_\mu\Sigma^a D^\mu\Sigma^a - \frac{m_\Sigma^2}{2}\Sigma^a\Sigma^a - \frac{b_4}{4}(\Sigma^a\Sigma^a)^2$$

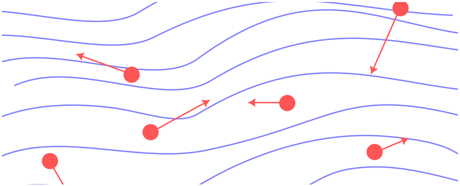
Niemi et al. '20', OG & Tenkanen '23'

Lattice for real-time physics

Hard thermal loops

- Generalises dimensional reduction for time-dependent quantities. Braaten, Pisarski, Frenkel, Taylor, Wong '90; Blaizot & Iancu '94

- Equivalent to fluctuating classical fields, coupled to fluctuating Boltzmann particles



$\rightarrow$ possible (but tricky) to put on the lattice.

e.g. Bödeker, Moore, Rummukainen '00

- For scalars, or gauge fields further in the IR, this becomes Langevin evolution Aarts & Smit '97; Bödeker '98; Greiner & Müller '00
$\rightarrow$ easy to put on the lattice.

Lattice simulations of bubble nucleation — a history

- Stochasic lattice simulations proposed for studying nucleation

  Grigoriev & Rubakov; Bochkarev & de Forcrand '88

- Lattice and semiclassical predictions strongly disagree on nucleation rate

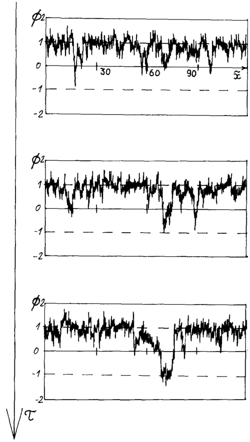
  Valls & Mazenko '90; Alford et al. '91

- Lattice counterterms and fitting improves qualitative agreement

  Alford et al. '93; Borsanyi et al. '00

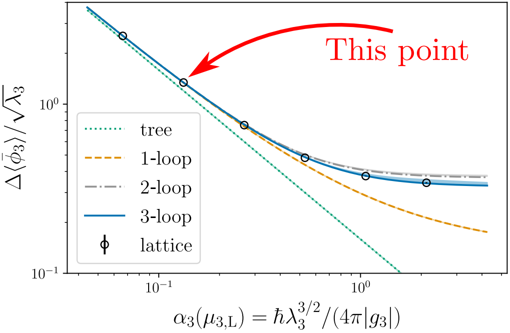
- New method for suppressed transitions, but still $e^{O(10)}$ to $e^{O(100)}$ disagreement

  Moore, Rummukainen, Tranberg '00, '01



Grigoriev & Rubakov '88

A super perturbative benchmark point



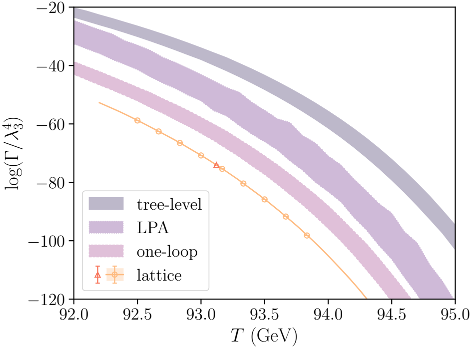
Perturbation theory converging *very* quickly for latent heat,

$$\underbrace{1.341(2)}_{\text{lattice}} \stackrel{?}{=} \underbrace{1.2}_{\text{tree}} + \underbrace{0.1378}_{\text{1-loop}} + \underbrace{0.0054}_{\text{2-loop}} - \underbrace{0.0016}_{\text{3-loop}} + \ldots$$

$$\stackrel{\checkmark}{=} 1.34170(4)$$

OG '21

Benchmarking against the lattice



Qualitative agreement for log rate, but *way* worse than latent heat,

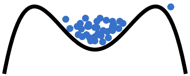
$$\underbrace{-74.09(5)}_{\text{lattice}} \overset{?}{=} \underbrace{-38.02}_{\text{tree}} - \underbrace{25.32}_{\text{1-loop}} + \dots$$

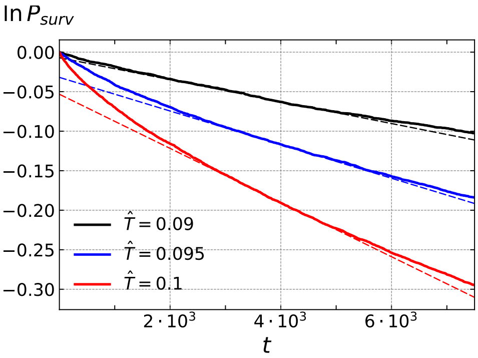
$$\overset{x}{=} -63(3)$$

OG, Kormu & Weir '24; see also Seppä et al. '25; Hällfors & Rummukainen '25

Nucleation in $d = 1 + 1$

$$H = \int \mathrm{d}x \left(\frac{\pi^2}{2} + \frac{(\partial_x \phi)^2}{2} + \frac{m^2 \phi^2}{2} - \frac{\lambda \phi^4}{4} \right)$$





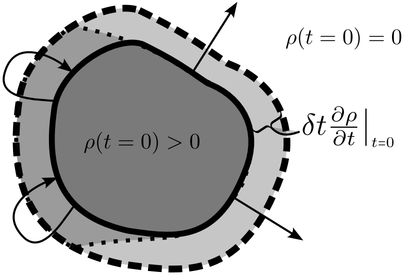
Wait-and-see simulations:

- → rate is time dependent
- → rate is lower than predicted
- → IR modes fail to thermalise

Pîrvu, Shkerin & Sibiryakov '24

see also Batini et al. '23

What about a fully thermalised metastable phase?



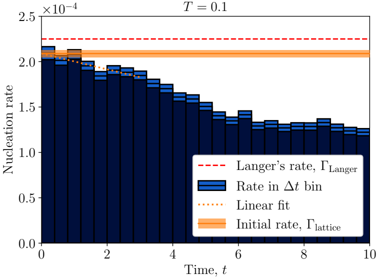
Define the nucleation rate as the initial escape rate,

$$\Gamma \equiv -\int_{\text{meta}} \mathcal{D}\phi\mathcal{D}\pi\ \delta_{\text{NR}}(\mathcal{T})\frac{\partial\rho}{\partial t}\bigg|_{t=0},$$

excluding trajectories that return in a microscopic timescale.

Hirvonen & OG '25

Langer's nucleation rate reproduced on the lattice



$$\Gamma_{\text{Langer}} = 2.25(23) \times 10^{-4}\,, \qquad \Gamma_{\text{lattice}}^{\text{thermal}} = 2.09(4) \times 10^{-4}\,.$$

A fully thermalised metastable phase nucleates with Langer's rate (at least in 1+1d). Hirvonen & OG '25

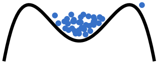
Summing up

Lattice simulations are important for:

- resolving phase diagrams (e.g. electroweak)
- studying confinement transitions
- testing perturbative approaches

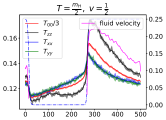
What we don't yet know:

- Is nucleation *fully thermal* in cosmology?
- Why is nucleation harder in 3+1d than 1+1d?



Some things we didn't cover:

- v_w on the lattice
- other nonperturbative methods



Mou et al. '20

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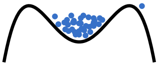

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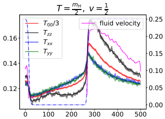
- v_w on the lattice
- other nonperturbative methods

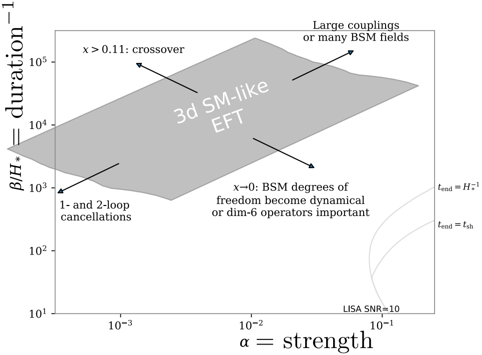


Mou et al. '20

Thanks for listening!

Backup slides

Mapping BSM models to the electroweak high-T EFT

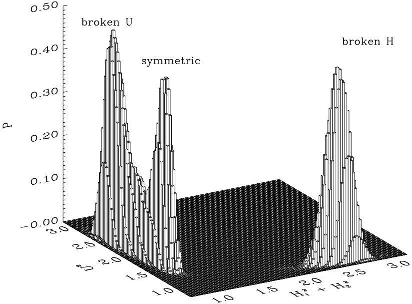


- EFT characterised by $x \equiv \lambda_3/g_3^2$, with $x_{\rm SM} \approx \frac{m_H^2}{8m_W^2}$.
- New scalar σ with Higgs-mixing angle θ modifies x as

$$x \approx x_{\rm SM} + \frac{m_\sigma^2 - m_H^2}{8m_W^2} \sin^2\theta$$

OG et al. '19

Simulations beyond the SM



- Minimally supersymmetric SM

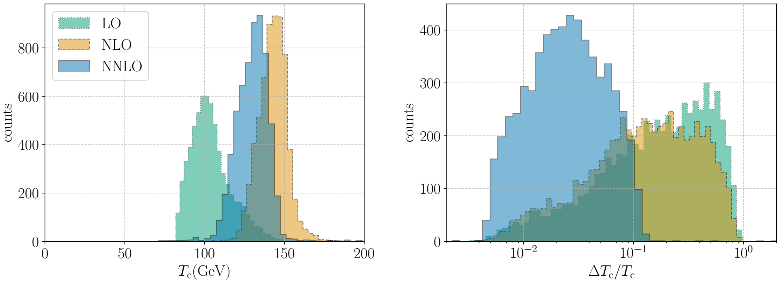
  Laine & Rummukainen '99, '00; Laine, Nardini & Rummukainen '12

- SM plus a new scalar singlet (xSM), doublet (2HDM) or triplet (ΣSM) Niemi et al. '19, '20, '24

- Simpler models without electroweak sector

  Karjalainen et al. '96; Arnold & Moore '01; Sun '02; Jüttner et al. 20; OG '21

Distribution shifts



$$\mathscr{L} = \mathscr{L}_{\rm SM} - \frac{a_1}{2}(\Phi^\dagger\Phi)S - \frac{a_2}{2}(\Phi^\dagger\Phi)S^2 - \frac{1}{2}(\partial S)^2 - \frac{m_S^2}{2}S^2 - \frac{b_3}{3}S^3 - \frac{b_4}{4}S^4$$

OG & Saffin '24